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# ON IDEALS AND QUOTIENTS OF HERMITIAN ALGEBRAS

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Abstract: We prove that a $*$-algebra $\mathcal{A}$ is hermitian if and only if a closed two-sided ideal I and the quotient $\mathcal{A}/I$ are hermitian.

Key words: Hermitian algebra, the Pták's pseudonorm.

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Let A be a complex Banach $*$-algebra possibly without unit. The spectral radius and Pták's function of the element $a \in A$ will be denoted respectively by $|a|_\sigma$ and p(a). Here by definition $p(a) = |a^* a|_\sigma^{1/2}$. The set of selfadjoint elements of A (i.e. such that $a^* = a$) is denoted by H(A). Let I be a selfadjoint closed ideal in A. Our purpose in this note is to prove the next theorem:

The algebra A is hermitian if and only if I and A/I are hermitian.

In the case of isometric involution this result has been recently obtained by H. Leptin [1].

The recent Pták's contribution to the theory of hermitian algebras [3] make it possible to prove the result in its full generality without any continuity assumption concerning the involution.

For the proof of the main theorem we need the following characterization of hermitian algebras.

Theorem 1. Let A be a Banach $*$-algebra. Then the following properties are equivalent.

$1^o$ A is hermitian.

$2^o$ For every proper left ideal $L \subset A$ there exists a non-zero positive linear functional $f$ with $f(L) = 0$.

$3^o$ For every proper modular left ideal $L \subset A$ there exists a non-zero positive linear functional $f$ with $f(L) = 0$.

Proof. Assume $1^o$. Set $L_1 = \{x + \lambda : x \in L, \lambda \text{ complex}\}$ so that $L_1$ is a linear subspace of $A_1$ (where $A_1 = \{a + \nu : a \in A, \nu \text{ complex}\}$, i.e. the unitization of A).

Now define $f_o(x + \lambda) = \lambda$ for each $x + \lambda \in L_1$. Then $f_o$ is a linear functional on $L_1$ with $f_o(1) = 1$. It is evident that L is a proper left ideal in $A_1$. Therefore, we have $0 \in \sigma(x)$ for all $x \in L$, hence $\lambda \in \sigma(x + \lambda)$. Hence $|f_o(x + \lambda)| = |\lambda| \leq |x + \lambda|_\sigma$ .

Since, by definition, A is hermitian if and only if $A_1$ is hermitian, we can use the fundamental inequality [3]. It follows

$$|f_o(x + \lambda)| \leq |x + \lambda|_\sigma \leq p_{A_1}(x + \lambda).$$

The Pták's function p being a pseudonorm on hermitian algebras [3], we can extend $f_o$, by Hahn-Banach extension theorem, to a linear functional f satisfying $|f(a)| \leq p(a)$ for all $a \in A_1$. Now by Theorem 6.4 of [3], f is state on $A_1$. By definition $f(L) = 0$.

In this fashion we have obtained the implication $1^o \longrightarrow 2^o$. The implication $2^o \longrightarrow 3^o$ is immediate.

Assume $3^{o}$ and let us prove $1^{o}$. Let $b \in A$ and $h = b^{*} b$. If $A(1 + h)$ would be a proper modular left ideal in $A$, then there would exist, by assumption, a non-zero positive definite linear functional $f$ with $f(A(1 + h)) = 0$.

It follows that $f(a + h) = f(a) + f(ah) = 0$ for all $a \in A$. Putting $a = h$, we obtain $f(h) + f(h^{2}) = 0$. The functional $f$ being positive, this implies $f(h) = 0 = f(h^{2})$. From the Cauchy-Schwartz inequality we conclude $f(ah) = 0$ for all $a \in A$ and hence $f(a) \equiv 0$, $f = 0$, which is not the case. Therefore $A(1 + h) = A$. This means that $-1 \notin \sigma(h)$ and so $A$ is hermitian. The proof is complete.

Remark. For locally continuous involution the implication $1^{o} \longrightarrow 3^{o}$ was proved in the monograph of C. Rickart [4, p. 236] and the implication $3^{o} \longrightarrow 1^{o}$ is due to H. Leptin [2].

Now using these results we can state our main

Theorem 2. Let $A$ be a Banach $*$-algebra. The algebra $A$ is hermitian if and only if $I$ and $A/I$ are hermitian.

Proof. Let $A$ be hermitian and let $I$ be a closed selfadjoint ideal of $A$. Then, it is well known that for each $x \in I$ the following relations hold:

$$\sigma_{A}(x) \subset \sigma_{I}(x)$$

and

$$\partial\sigma_{I}(x) \subset \partial\sigma_{A}(x)$$

where $\partial$ stands for the boundary of the spectrum.

Now, if $x \in H(I)$ then we have the following inclusions: $\sigma_{A}(x) \subset R^{1}$ and $\partial\sigma_{I}(x) \subset \partial\sigma_{A}(x) \subset R^{1}$. It follows that $\sigma_{I}(x) \subset R^{1}$, i.e. $I$ is hermitian.

Now denote by $\pi$ the canonical quotient $*$-homomorphism of A modulo I, i.e. $\pi : A \longrightarrow A/I$. It is well known that $\sigma_{A/I}(\pi(a)) \subset \sigma_A(a)$ for any $a \in A$.

Let $\pi(x)^* = \pi(x)$. Then there exists $z \in \pi(x)$, which is in H(A). Hence $\sigma_{A/I}(\pi(x)) = \sigma_{A/I}(\pi(z)) \subset \sigma_A(z) \subset R^1$, i.e. A/I is hermitian.

Conversely, assume I and A/I are hermitian and show that any maximal modular left ideal L in A is annihilated by some non-zero positive functional f on A. Let u be a unit modulo L.

Without restriction of generality, we assume $I \neq A$. We consider first the case when $A \neq I + L$. Then $M = I + L$ is a proper modular left ideal in A hence the set $\pi(M)$ is a left ideal in A/I. We show that $\pi(M)$ is proper. Indeed, if $\pi(u) \in \pi(M)$ then $u - m \in I \subset M$ for some $m \in M$ so that $u \in M$, which is a contradiction.

Thus $\pi(M)$ is a proper left ideal in the hermitian algebra A/I. Hence there exists a non-zero positive functional F on A/I such that $F(\pi(M)) = 0$. We define a non-zero positive functional f on A by $f(a) = F(\pi(a))$.

Obviously $f(M) = 0$. This proves the first case.

It remains the case when $A = I + L$. Then $L_0 = I \cap L$ is a proper left ideal in I, hence there exists a non-zero positive functional $f_0$ on I with $f_0(L_0) = 0$. If $a = j_1 + \ell_1 = j_2 + \ell_2$ with $j_1, j_2 \in I$ and $\ell_1, \ell_2 \in L$ then $j_1 - j_2 = \ell_2 - \ell_1 \in L_0$. Hence $f_0(j_1) = f_0(j_2)$. So we can extend $f_0$ to the whole of A in the natural way: for $a = j + \ell$ with $j \in I$, $\ell \in L$, put $f(a) = f_0(j)$. Hence $f(L) = 0$. Obviously

$f$ is a non-zero functional and we show only that it is positive.

We have $a^* a = j^* j + \ell^* j + a^* \ell$. Here $a^* \ell \in L$, so that $f(a^* \ell) = 0$. To compute $f(\ell^* j)$, we observe that $\ell^* j \in I$ whence $f(\ell^* j) = f_0(\ell^* j)$. Since $f_0$ is positive, we have $f_0(\ell^* j) = (f_0(j^* \ell))^*$, but $f_0(j^* \ell) = 0$ since $j^* \ell \in L_0$.

Thus $f(a^* a) = f_0(j^* j) \geq 0$ and the proof is complete.

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