COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

18,1 (1977)

ITERATED ULTRAPOWER AND PRIKRY'S FORCING

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Abstract: It is shown that the factorization of the Boolean ultrapower ${}^{B}V$ by a suitable ultrafilter $\overline{\mathcal{U}}$ is isomorphic to the Gaifman's direct limit of the iterated ultrapowers $\mathcal{N}_n, n \in \omega_0$, where B is the Boolean algebra of the Prikry's forcing. Moreover, the corresponding extension $V^{(B)}/\overline{\mathcal{U}}$ is isomorphic to the intersection $\bigcap_{n \in \omega_0} \mathcal{N}_n$.

Key words: Iterated ultrapower, generic extension, forcing, measurable cardinal.

AMS: 02K05, 02K35 Ref. Ž.: 2.641.3

In the note [1] I have shown that the intersection $\mathcal{N} = \bigcap_{n \in \omega_0} \mathcal{N}_n$ of n-th ultrapowers $\mathcal{N}_n$ of the universe V is a generic extension of the Gaifman's direct limit $\mathcal{N}_{\omega_0}$ of $\mathcal{N}_n, n \in \omega_0$ (with corresponding elementary embeddings). Moreover, this generic extension possesses properties similar to those of the extension constructed by K. Prikry [4]. P. Dehornoy [2] has proved that actually $\mathcal{N}$ is the generic extension of $\mathcal{N}_{\omega_0}$ by Prikry's forcing (constructed inside the model $\mathcal{N}_{\omega_0}$.). In this note I will prove the same result

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x) The result of this note has been presented on the Logic Colloquium, Clermont-Ferrand 1975.

by a method different from that of P. Dehornoy and obtain some additional information. Namely, I will prove the following theorem:

Let κ be a measurable cardinal, $\mathcal{U}$ being a normal measure on κ. Let B denote the complete Boolean algebra constructed from the Prikry's forcing. Let $\overline{\mathcal{U}}$ be the ultrafilter on B constructed from $\mathcal{U}$ by (3). Then

i) the ultrapower ${}^{B}V / \overline{\mathcal{U}}$ is isomorphic to the model class $\mathcal{N}_{\omega_0}$,

ii) the factorization $V^{(B)}/\overline{\mathcal{U}}$ of the Boolean-valued model $V^{(B)}$ is isomorphic to the intersection $\bigcap_{n \in \omega_0} \mathcal{N}_n$ and

iii) $\bigcap_{n \in \omega_0} \mathcal{N}_n = \mathcal{N}_{\omega_0}[a]$, where the set a is a generic subset of the Prikry's forcing.

Terminology and notations are those of [1] and [3]. Howwever we remind some of them here.

If C is a complete Boolean algebra, ${}^{C}V$ will denote the class of all functions f such that the domain $\mathcal{D}(f)$ of f is a partition of C (i.e. elements of $\mathcal{D}(f)$ are pairwise disjoint and the union of $\mathcal{D}(f)$ is 1). For any formula φ of the language of the set theory, one can define the Boolean value

$$|\varphi(f_1,\dots,f_n)|_C \in C$$

$f_1,\dots,f_n \in {}^{C}V$, in the obvious way, e.g.

$$|f_1 \in f_2|_C = \bigvee\{x \in C;\ (\exists u)(\exists v)(x \leq u \,\&\, x \leq v \,\&\, f_1(u) \in f_2(v))\}.$$

If $\mathcal{V}$ is an ultrafilter on C, we obtain the Boolean ultrapower ${}^{C}V / \mathcal{V}$ defining the membershiprelation $\in_{\mathcal{V}}$ as follows:

$$f \in_{\mathcal{V}} g \equiv | f \in g |_C \in \mathcal{V} .$$

The famous Łoś-theorem says that

(1) $\quad {}^C V / \mathcal{V} \models \varphi(f_1,\ldots,f_n) \equiv | \varphi(f_1,\ldots,f_n) |_C \in \mathcal{V}.$

The Boolean-valued model $V^{(C)}$ and the Boolean value $\| \varphi(f_1,\ldots,f_n) \|_C \in C$ are defined e.g. in [3]. If the ultrafilter $\mathcal{V}$ is σ-additive, then one can define the interpretation $i_{\mathcal{V}}$ of $V^{(C)}$ as in [3], p. 58, by induction

$$i_{\mathcal{V}}(f) = \{ i_{\mathcal{V}}(g);\ \| g \in f \|_C \in \mathcal{V} \} .$$

Let $x \in V$. We set

$$\mathcal{D}(\hat{x}) = \{1\}, \quad \hat{x}(1) = x.$$

Then $\hat{x} \in {}^C V$. The function $\check{x} \in V^{(C)}$ is defined in [3], p. 53.

If $\mathcal{V}$ is σ-additive, then ${}^C V / \mathcal{V}$ is well-founded and there exists an isomorphism $\psi_{\mathcal{V}}$ of ${}^C V / \mathcal{V}$ onto a transitive class. One can easily define an embedding χ of ${}^C V$ into $V^{(C)}$ such that

$$\chi(\hat{x}) = \check{x}$$

for any $x \in V$. It is easy to see that for any $f \in {}^C V$, the following holds true:

(2) $$i_{\mathcal{V}}(\chi(f)) = \psi_{\mathcal{V}}(f).$$

Let κ be a measurable cardinal, $\mathcal{U}$ being a normal measure on κ . $\mathcal{U}_n$ denotes the ultrafilter $\mathcal{U} \times \ldots \times \mathcal{U}$ on κ^n. The set P of Prikry's conditions is defined as follows (n = 0 is allowed!):

$p \in P \equiv p = \langle\langle \alpha_1, \dots, \alpha_m \rangle, X \rangle; \alpha_1 < \dots < \alpha_m < \inf X, X \in \mathcal{U}$,

$p' \leq p'' \equiv m' \geq m'', \alpha'_1 = \alpha''_1, \dots, \alpha'_{m''} = \alpha''_{m''}, X' \subseteq X'', \alpha' \in X''$

for $n'' < i \leq n'$.

Let B denote the complete Boolean algebra containing $P, \leq$ as a dense subset. We define $\overline{\mathcal{U}} \subseteq B$ by the condition

(3) $A \in \overline{\mathcal{U}} \equiv (\exists X \subseteq \kappa)(X \in \mathcal{U} \;\&\; \langle \emptyset, X \rangle \leq A)$.

K. Prikry [5] has proved that $\overline{\mathcal{U}}$ is a κ-complete ultrafilter on B.

Let $\kappa^{(n)}$ denote the set $\{\langle \xi_1, \dots, \xi_m \rangle; \xi_1 < \dots < \xi_m \in \kappa\}$. Evidently $\kappa^{(n)} \in \mathcal{U}_m$. For $\langle \xi_1, \dots, \xi_m \rangle \in \kappa^{(m)}$, we set

$$p_{\xi_1, \dots, \xi_m} = \langle\langle \xi_1, \dots, \xi_m \rangle; \kappa - (\xi_m + 1) \rangle .$$

The set $\{p_{\xi_1, \dots, \xi_m}; \langle \xi_1, \dots, \xi_m \rangle \in \kappa^{(m)}\}$ is a partition of the Boolean algebra B. By a simple computation one can prove for each $X \subseteq \kappa^{(n)}$ that

(4) $\bigvee_{\langle \xi_1, \dots, \xi_m \rangle \in X} p_{\xi_1, \dots, \xi_m} \in \overline{\mathcal{U}} \equiv X \in \mathcal{U}_m$.

The set

$$B_n = \{ \bigvee_{\langle \xi_1, \dots, \xi_m \rangle \in X} p_{\xi_1, \dots, \xi_m} ; X \subseteq \kappa^{(m)} \}$$

is a complete subalgebra of the Boolean algebra B. Evidently $B_n \subseteq B_{n+1}$. Moreover, B_n is atomic with the set $\{p_{\xi_1, \dots, \xi_m} ; \langle \xi_1, \dots, \xi_m \rangle \in \kappa^{(m)}\}$ of atoms. Since $\kappa^{(n)} \in \mathcal{U}_n$, the mapping φ_n defined for $f \in {}^{B_m}V$ as

$$\varphi_m(f)(\langle \xi_1, \dots, \xi_m \rangle) = f(p_{\xi_1, \dots, \xi_m})$$

induces an isomorphism - denoted by the same letter $\wp_n$ - of ${}^{B_n}V/\overline{\mathcal{U}}$ onto ${}^{\kappa^n}V/\mathcal{U}_n$.

The inclusion $B_n \subseteq B_m$, $n \leq m$ induces the natural embedding

$$ {}^{B_n}V/\overline{\mathcal{U}} \subseteq {}^{B_m}V/\overline{\mathcal{U}} \; . $$

Let $j_{n,m}$ denote the natural embedding of ${}^{\kappa^n}V/\mathcal{U}_n$ into ${}^{\kappa^m}V/\mathcal{U}_m$. The transitive class $\mathcal{N}_n$ is equal to $\psi_{\mathcal{U}_n}({}^{\kappa^n}V/\mathcal{U}_n)$ and $\nu_{n,m}$ is the corresponding embedding of $\mathcal{N}_n$ into $\mathcal{N}_m$.
Since $B_n \subseteq B$, we can write

$$ {}^{B_n}V/\overline{\mathcal{U}} \subseteq {}^{B}V/\overline{\mathcal{U}} \; . $$

We show that in fact

$$ (5) \qquad {}^{B}V/\overline{\mathcal{U}} = \bigcup_{n \in \omega_0} {}^{B_n}V/\overline{\mathcal{U}} \; . $$

Let $f \in {}^{B}V$. We can assume that $\mathcal{D}(f) \subseteq P$, i.e. that f is defined on the elements of P. Let

$$ P_n = \{ \langle\langle \xi_1, \dots, \xi_m \rangle, X \rangle \in P \; ; \; m = n \} \; . $$

Then $\mathcal{D}(f) = \bigcup_n (\mathcal{D}(f) \cap P_n)$ and also

$$ 1 = \bigvee \mathcal{D}(f) = \bigvee_n \bigvee (\mathcal{D}(f) \cap P_n). $$

Since $\overline{\mathcal{U}}$ is σ-additive, there exists a natural number n such that

$$ \bigvee(\mathcal{D}(f) \cap P_n) \in \overline{\mathcal{U}} \; . $$

If $p,q \in P_n \cap \mathcal{D}(f)$, $p \neq q$ then $p \wedge q = 0$. Thus, if $p = \langle\langle \xi_1, \dots, \xi_m \rangle, X \rangle$, $q = \langle\langle \eta_1, \dots, \eta_m \rangle, Y \rangle$, then

$\langle \xi_1,\dots,\xi_m \rangle \neq \langle \eta_1,\dots,\eta_m \rangle$ (since otherwise $\langle\langle \xi_1,\dots,\xi_m \rangle, X \cap Y \rangle \leq p \wedge q$). We define $\bar{f} \in {}^{B_m}V$ as follows:

$$\bar{f}(p_{\xi_1,\dots,\xi_m}) = f(\langle\langle \xi_1,\dots,\xi_m \rangle, X \rangle) \text{ if}$$

$$\langle\langle \xi_1,\dots,\xi_m \rangle, X \rangle \in \mathcal{D}(f) \cap P_m \quad \text{for some } X,$$

$$= 0 \text{ otherwise.}$$

Evidently $\bar{f} \in {}^{B_m}V$ and

$$|f = \bar{f}|_B \geq \bigvee (P_n \cap \mathcal{D}(f)) \in \bar{\mathcal{U}} .$$

From the definition of the direct limit of the system ${}^{\kappa^n}V/\mathcal{U}_m, j_{m,m}$ and from (5) we obtain a natural isomorphism φ_{ω_0} from ${}^{B}V/\bar{\mathcal{U}}$ onto $\lim\limits_{m} {}^{\kappa^n}V/\mathcal{U}_m$.

If ψ_{ω_0} is the isomorphism of $\lim\limits_{m} {}^{\kappa^n}V/\mathcal{U}_n$ onto the transitive class $\mathcal{N}_{\omega_0}$, then $\varphi_{\omega_0} \circ \psi_{\omega_0}$ is an isomorphism from ${}^{B}V/\bar{\mathcal{U}}$ onto $\mathcal{N}_{\omega_0}$.

Since the interpretation $i_{\bar{\mathcal{U}}}$ of the model $V^{(B)}/\bar{\mathcal{U}}$ maps the submodel ${}^{B}V/\bar{\mathcal{U}}$ (more precisely, the submodel $\chi({}^{B}V)/\bar{\mathcal{U}}$ onto a transitive class, one can easily see that

(6) $$\chi \circ i_{\bar{\mathcal{U}}} = \varphi_{\omega_0} \circ \psi_{\omega_0} ,$$

i.e., for $f \in {}^{B}V$ we have

$$i_{\bar{\mathcal{U}}}(\chi(f)) = \psi_{\omega_0}(\varphi_{\omega_0}(f)) \in \mathcal{N}_{\omega_0} .$$

Let $h \in V^{(B)}$ be such that

$$\| h(\check{n}) = \check{\xi} \| = \bigvee_{\xi_1 < \dots < \xi_{m-1} < \xi} p_{\xi_1,\dots,\xi_{m-1},\xi} .$$

By an easy computation we obtain

(7) $$i_{\overline{\mathcal{U}}}(h) = \{ \langle n, \kappa_n \rangle ; n \in \omega_0 \}$$

where $\kappa_n = \psi_{\mathcal{U}_n}(\hat{\kappa})$ is the measurable cardinal in $\mathcal{N}_n$.

By K. Prikry [4], the generic extension $V^{(B)}/\overline{\mathcal{U}}$ of ${}^{B}V/\overline{\mathcal{U}}$ (more precisely, of $\times({}^{B}V)/\overline{\mathcal{U}}$) is such that $V^{(B)}/\overline{\mathcal{U}} = {}^{B}V/\overline{\mathcal{U}}\,[h]$. Thus, by (6) and (7) we obtain

(8) $$i_{\overline{\mathcal{U}}}(V^{(B)}/\overline{\mathcal{U}}) = \mathcal{N}_{\omega_0}[\{\langle n, \kappa_n \rangle ; n \in \omega_0\}] .$$

In [1], we have proved, denoting $\{\langle n, \kappa_n \rangle ; n \in \omega_0\}$ by a, that

$$\mathcal{N}_n \supseteq \mathcal{N}_{\omega_0}[a]$$

for every $n \in \omega_0$. Now, we shall show that also

(9) $$\bigcap_n \mathcal{N}_n = \mathcal{N}_{\omega_0}[a] .$$

Using the theorem of R. Balcar and P. Vopěnka, [3], p. 38, it suffices to show that each $x \in \bigcap_n \mathcal{N}_n$, $x \subseteq On$ is an element of the class $\mathcal{N}_{\omega_0}[a]$.

We set

$$x_n = \{ \xi ; \nu_{n,\omega_0}(\xi) \in X \} .$$

Then

$$\nu_{n,\omega_0}(x_n) \subseteq x.$$

Let $f_n \in {}^{B_n}V$ be such that $\psi_{\mathcal{U}_n}(\varphi_n(f_n)) = x_n$. One can easily construct functions $g_n \in {}^{B_n}V$, $n \in \omega_0$ in such a way that $f_1 = g_1$ and

$$|\hat{\xi} \in g_{n+1}|_{B_{n+1}} = |\hat{\xi} \in g_n|_{g_n} \vee |\hat{\xi} \in f_{n+1}|_{B_{n+1}} .$$

By simple computation we have

$$\psi_n(\varphi_n(g_n)) = x_n.$$

Now, we define $f \in V^{(B)}$ as follows:

$$f(\hat{\xi}) = \bigvee_n |\hat{\xi} \in g_n|_{B_n} \in B .$$

Then

$$i_{\overline{\mathcal{U}}}(f) = x,$$

thus, by (8),

$$x \in \mathcal{N}_{\omega_0}[a] .$$

Let us remark that the model $V^{(B)}/\overline{\mathcal{U}}$ is well-founded, but $\overline{\mathcal{U}}$ is not generic ultrafilter. In fact, the existence of such a non-trivial well-founded (Boolean) model implies the existence of a measurable cardinal.

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(Oblatum 15.11.1976)