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ONE EXAMPLE CONCERNING TESTING CATEGORIES

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Abstract: It is shown that there is a complete, co-complete, extremally well- and co-well-powered category A which contains any one-object category as a full subcategory, but there is a small category not equivalent to a full subcategory of A.

Key words: Testing category, Mac Neille completion.

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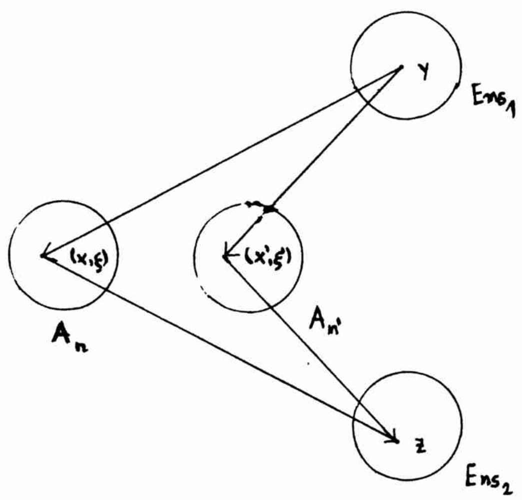
The result stated in the abstract answers a question which naturally arises in the study of testing categories. Namely, under a mild set-theoretic assumption there is a two-object category full embeddability of which into a complete and extremally well-powered category A make any concrete category to be equivalent with a full subcategory of A. Further, for any set S of one-object categories there is a complete, co-complete, well- and co-well-powered category A which contains any category from S as a full subcategory, but there is a small category not equivalent to a full subcategory of A (see [3]). The last example is constructed by means of a suitable completion of a coproduct of categories from S. I did not succeed in managing so with all one-object categories. But one can make use of the Mac Neille completion of a faithful functor in the sense of [1]. The point of it is that the corresponding "Mac Neille completion" of a category C, i.e. a completion in which C is dense and codense almost never exists (see [2]). I hint at the fact that the category A which will be constructed is neither well-powered nor co-well-powered. It remains a question whether it is possible. A disadvantage of A is that it is not fibre small (A has a proper class of non-isomorphic structures on each underlying set x).

Let N be a category which has as components all one-object categories and U: N ⟶ Ens be a functor such that the restriction of U on an object n of N is the hom-functor N(n,-). Let V: A ⟶ Ens be the Mac Neille completion of U. Then A looks as follows:



Here Ens_1 and Ens_2 are copies of the category of sets and A_n are indexed by objects of N. Objects of A_n are couples (x, ς) where x is a set and ς is a certain set of mappings $x \longrightarrow Un$ such that the following condition is satisfied: If η_ς is the set of all mappings $g: Un \longrightarrow x$ such that for each $f \in \varsigma$ there is a morphism $h: n \longrightarrow n$ in N such that $Uh = fg$, then ς is the set of all mappings $f: x \longrightarrow Un$ such that for each $g \in \eta_\varsigma$ there is $h: n \longrightarrow n$ in N such that $Uh = fg$.

Morphisms $(x, \varsigma) \longrightarrow (x', \varsigma')$ in A_n are mappings $f: x \longrightarrow x'$ such that $gf \in \varsigma$ for each $g \in \varsigma'$. If $n \neq n'$, then there is no morphism between objects of A_n and $A_{n'}$. Let $y \in Ens_1$, $z \in Ens_2$ and $(x, \varsigma) \in A_n$. Then morphisms $y \longrightarrow (x, \varsigma)$ and $(x \varsigma) \longrightarrow z$ are mappings $y \longrightarrow x$ and $x \longrightarrow z$. So there is no morphism from Ens_2 to A_n and from A_n to Ens_1. Morphisms in A compose as mappings and V is the obvious underlying functor.

It is easy to show that A is complete and cocomplete and that V preserves limits and colimits (after all it follows from [1]). Thus each category A_n is well- and co-well-powered. Let $y \in Ens_1$, $(x, \varsigma) \in A_n$ and $z \in Ens_2$. Any morphism $f: y \longrightarrow (x, \varsigma)$ can be factorized as $y \xrightarrow{f} x \xrightarrow{1_x} (x, \varsigma)$ and similarly any $g: (x, \varsigma) \longrightarrow z$ as $(x, \varsigma) \xrightarrow{1_x} x \xrightarrow{g} z$. Hence f cannot be extremally epi and g extremally mono. Thus A is extremally well- and co-well-powered.

Following [1] there is a full embedding $Y: N \longrightarrow A$. It suffices to put $Yn = (Un, \{Uf/f: n \longrightarrow n\})$ and $Yh = Uh$. Let $(x, \varsigma) \in A_n$, $f \in \varsigma$ and $g \in \eta_\varsigma$. Then $f: (x, \varsigma) \longrightarrow Yn$ and

$g: Yn \longrightarrow (x, \varsigma)$ are morphisms in A_n. So for any $(x, \varsigma) \in A_n$ such that $\emptyset \neq \varsigma \neq (Un)^x$ there are morphisms $Yn \longrightarrow (x, \varsigma) \rightarrow$ $\longrightarrow Yn$.

Suppose that the following category is a full subcategory of A (there are indicated non-identical morphisms)

$$\begin{tikzcd} a \arrow[loop above] \arrow[dr, shift left] \arrow[dr, shift right] & & b \arrow[loop above] \arrow[dl] \\ & c & \end{tikzcd}$$

Since a, b have exactly two endomorphisms, they differ from objects of the type $(x, \emptyset)$ or $(x, (Un)^x)$. Hence a, b do not belong to the same A_n because otherwise it would be a morphism $a \longrightarrow Yn \longrightarrow b$. Thus $c \in Ens_2$. Since c has exactly one endomorphism, c equals to $\emptyset$ or 1. But now one cannot have two morphisms from a to c.

We have shown that A has the desired properties.

References

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