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ON THE LOCAL ERGODIC THEOREMS OF KRENGEL, KUBOKAWA, AND TERRELL x)

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Abstract: Let (X, Σ, μ) be a σ-finite measure space and $\Gamma = \{T_t : t \geq 0\}$ a strongly continuous semigriup of positive $L_p(\mu)$ operators, $1 \leq p < \infty$. We present direct proofs of Krengel's and Kubokawa's local ergodic theorems using a method which easily extends to the case of n-parameter semigroups. The result obtained in the n-parameter case generalizes a theorem of T.R. Terrell.

Key words: Local ergodic theorem, semigroups of positive L_p operators, strongly continuous semigroups, n-parameter semigroups.

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Introduction. Let (X, Σ, μ) be a σ-finite measure space and $1 \leq p \leq \infty$. Denote by $L_p(\mu)$ the usual Banach spaces of complex-valued functions. Let $\Gamma = \{T_t : t \geq 0\}$ be a strongly continuous semigroup of bounded $L_p(\mu)$ operators. This means that (i) $\| T_t \|_p < \infty$, $t \geq 0$; (ii) $T_{s+t} = T_s T_t$; (iii) $\| T_t f - T_s f \| \longrightarrow 0$ as $t \longrightarrow s$ for all $f \in L_p(\mu)$. We say that T_t is positive if $f \in L_p^+(\mu) \Longrightarrow$

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$\Longrightarrow T_t f \in L_p^+(\mu)$ and that T_t is a <u>contraction</u> if $\| T_t \| \leq 1$. We assume $T_0 = I$, although the results obtained hold when $T_0 \neq I$ if appropriate modification is made. For a strongly continuous $L_p(\mu)$ semigroup Γ and $f \in L_p(\mu)$, we set

$$A(\varepsilon,T)f(x) = (1/\varepsilon) \int_0^\varepsilon T_t f(x)dt$$

for $\varepsilon > 0$. We say that the <u>local</u> <u>ergodic</u> <u>theorem</u> (L.E.T.) holds for Γ if

$$\lim_{\varepsilon \to 0+} A(\varepsilon,T)f(x) = f(x) \quad \text{a.e.}$$

for $f \in L_p(\mu)$. Let us clarify the definition of $A(\varepsilon,T)f(x)$. The strong continuity of Γ ensures that the vector-valued function $t \longrightarrow T_t f$ is Lebesgue integrable over any finite interval (a,b) , $0<a<b<\infty$. It follows [4, p.196] that for each $f \in L_p(\mu)$, $1 \leq p < \infty$, there exists a scalar function $g(t,x)$ on $[0,\infty) \times X$, measurable with respect to the usual product σ-field, which is uniquely determined up to a set of measure zero in this space by the conditions:)i) for a.e.t. ≥ 0 , $g(t,.)$ belongs to the equivalence class of $T_t f$, (ii) there exists a μ-null set $E(f)$, independent of t , such that $x \notin E(f)$ implies $g(.,x)$ is Lebesgue integrable over every (a,b) , $0<a<b<\infty$, and $\int_a^b g(t,x)dt$ belongs to the equivalence class of $\int_a^b T_t f dt$. The function $g(t,x)$ is called the scalar representation of $T_t f$. We define $T_t f(x) = g(t,x)$. This justifies the definition of $A(\varepsilon,T)f(x)$. We note that for $x \notin E(f)$, $A(\varepsilon,T)f(x)$ is a continuous function of $\varepsilon > 0$.

In [6] U. Krengel established the L.E.T. for Γ a strongly continuous semigroup of positive $L_1(\mu)$ contractions. This result was obtained independently by D.S. Ornstein [11]. T.R. Terrell [12] generalized the Krengel-Ornstein theorem to the case of n-parameter semigroups of positive $L_1(\mu)$ contractions, $n > 1$. The proofs given were indirect and Terrell's method was completely different from Krengel's. In [7, 8], Y. Kubokawa extended Krengel's theorem to the case where Γ is a semigroup of positive $L_p(\mu)$ operators for some $1 \leq p < \infty$. His proofs depended on a maximal ergodic inequality for Γ which he derived in [7]. In this paper we obtain the theorems of Krengel, Ornstein, and Kubokawa by direct proofs utilizing a technique which easily extends to the n-parameter case. The n-parameter result obtained generalizes Terrel's theorem. We remark that M.A. Akcoglu and R.V. Chacon [2] proved the L.E.T. for the case when Γ is a semigroup of positive $L_1(\mu)$ contractions without assuming Γ to be strongly continuous at $t = 0$. Also Kubokawa [9] established the L.E.T. for Γ a semigroup of not necessarily positive $L_1(\mu)$ contractions.

Main results. We establish two preliminary lemmas. In [13, p.232] it is shown that if Γ is a strongly continuous semigroup then there exist $M > 0$, $a \geq 0$ such that $\| T_t \| \leq Me^{at}$, $t \geq 0$. For $T_t \in \Gamma$ we set $S_t = e^{-bt}T_t$ for some fixed $b > a$. We assume henceforth that all semigroups are strongly continuous for all $t \geq 0$.

1. Lemma. Let Γ be a semigroup of positive $L_p(\mu)$

operators for some $1 \le p < \infty$. For $f \in L_p(\mu)$ the following are equivalent:

(i) $\lim_{\varepsilon \to 0+} A(\varepsilon, T)f(x) = f(x)$ a.e.

(ii) $\lim_{\varepsilon \to 0+} A(\varepsilon, S)f(x) = f(x)$ a.e.

Proof. It is sufficient to establish the lemma assuming $f \in L_p^+(\mu)$. Suppose (i) holds. Since $S_t f(x) \le T_t f(x)$ for all $t \ge 0$, we have

$$\lim_{\varepsilon \to 0+} \sup A(\varepsilon, S)f(x) \le f(x) \quad \text{a.e.}$$

Given $0 < \varepsilon < 1$ there exists $\delta > 0$ sufficiently small that $e^{-bt} > 1 - \varepsilon$ for $0 < t < \delta$. Then

$$\lim_{\alpha \to 0+} \inf A(\alpha, S)f(x) \ge (1 - \varepsilon)f(x) \quad \text{a.e.}$$

Since ε is arbitrary we have $\lim_{\alpha \to 0+} \inf A(\alpha, S)f(x) \ge f(x)$ a.e. Thus $\lim_{\varepsilon \to 0+} A(\varepsilon, S)f(x)$ exists and equals $f(x)$ a.e. So (i) $\Longrightarrow$ (ii). Thus proof of the converse is similar. Q.E.D.

2. <u>Lemma</u>. Let Γ be a semigroup of positive $L_p(\mu)$ operators for some $1 < p < \infty$. There exists $0 < h \in L_p(\mu)$ such that if we set $m(A) = \int_A h^p d\mu$, $A \in \Sigma$, and $P_t f = S_t(fh)/h$ for $f \in L_p(m)$ then P_t can be extended by continuity to an $L_1(m)$ operator and $\Gamma' = \{P_t : t \ge 0\}$ becomes a strongly continuous semigroup of positive $L_1(m)$ contractions.

<u>Proof</u>. We only sketch the proof since the lemma appears in [10]. Since $\{S_t^*\}$ is a weakly continuous semigroup on $L_q(\mu)$, where $q = p/(p-1)$, and $L_q(\mu)$ is reflexive, it follows that $\{S_t^*\}$ is strongly continuous. Set $g = \int_0^\infty S_t^* g' dt$ for some $0 < g' \in L_q(\mu)$. Then $g \in L_q(\mu)$,

$g > 0$ a.e., and $S_t^* g \leq g$ for all $t \geq 0$. Set $h = g^{1/(p-1)}$. Then Γ' is a strongly continuous semigroup of positive $L_p(m)$ operators. Since $P_t^*(1) = S_t^*(g)/g \leq 1$ it follows that $\| P_t^* \|_\infty \leq 1$ and, consequently, that P_t can be extended by continuity to an $L_1(m)$ contraction. It is easy to show that Γ', regarded as an $L_1(m)$ semigroup, is strongly continuous. Q.E.D.

We remark that the lemma in [10] is more general than our lemma 2 in that it includes the case $p = 1$. It seems best in this paper to consider the case $p = 1$ in the proof of theorem 4 since by so doing it is easier to see how to prove the n-parameter extension of this theorem for the case $p = 1$. We now prove Krengel's and Ornstein's L.E.T. and then use it to establish Kubokawa's theorems.

3. <u>Theorem</u> (Krengel,Ornstein). Let Γ be a semigroup of positive $L_1(\mu)$ contractions. Then the L.E.T. holds for Γ.

<u>Proof</u>. For $0 < g \in L_1(\mu)$ set $h = \int_0^\infty S_t g(x)\,dt$. Then $0 < h \in L_1(\mu)$ and

$$S_t h = \int_t^\infty S_r g \, dr \leq \int_0^\infty S_r g \, dr = h(x)$$

for all $t \geq 0$. Setting $m(A) = \int_A h \, d\mu$ and $P_t f = S_t(fh)/h$ for $f \in L_1(m)$, we have $P_t(1) \leq 1$ a.e. which implies $\| P_t \|_\infty \leq 1$, $t \geq 0$. It is easy to check that $\| P_t \|_1 \leq 1$ also. By [4, p. 691] we have for $f \in L_1(m)$ and $\beta > 0$,

$$m\{f^* > \beta\} \leq (K/\beta) \int | f | \, dm \quad ,$$

where $f^* = \sup_{\varepsilon > 0} | A(\varepsilon, P)f(x)|$ and $K > 0$ is independent of f and $\Gamma' = \{P_t\}$. We see therefore that $f^* < \infty$ a.e. Now the class $M = \{A(\varepsilon, P)f : 0 < \varepsilon < 1 , f \in L_1(m)\}$ is dense in $L_1(m)$ and the L.E.T. holds for Γ' if $f \in M$ [6]. As was noted above, $A(\varepsilon, P)f(x)$ depends continuously on ε for x outside some null set. Thus it follows from Banach's convergence principle [4, p. 332] that $\lim_{\varepsilon \to 0+} A(\varepsilon, P)f(x)$ exists and is finite a.e. on X for all $f \in L_1(m)$. Since $A(\varepsilon, P)f \longrightarrow f$ in norm, we have $\lim_{\varepsilon \to 0+} A(\varepsilon, P)f(x) = f(x)$ for $f \in L_1(m)$.

It is easy to check that for $f \in L_1(\mu)$ and $\varepsilon > 0$ we have $A(\varepsilon, S)f = [A(\varepsilon, P)(f/h)] \cdot h$. Thus by lemma 1,

$$\begin{aligned}\lim_{\varepsilon \to 0+} A(\varepsilon, T)f(x) &= \lim_{\varepsilon \to 0+} A(\varepsilon, S)f(x) \\ &= \lim_{\varepsilon \to 0+} [A(\varepsilon, P)(f/h)(x)] \cdot h(x) \\ &= (f/h)h(x) \\ &= f(x) \quad \text{a.e.} \qquad \text{Q.E.D.}\end{aligned}$$

4. <u>Theorem</u> (Kubokawa). Let Γ be a semigroup of positive $L_p(\mu)$ operators for spme $1 \leq p < \infty$. Then the L.E.T. holds for Γ.

Proof. Assume first that $p > 1$. Then by lemma 2 and theorem 3 we have the L.E.T. holding for the $L_1(m)$ semigroup $\Gamma' = \{P_t\}$, where P_t is defined as in lemma 2. For $f \in L_p(\mu)$ we have $f/h \in L_p(m) \subset L_1(m)$ since $m(X) < \infty$. Thus

$$\begin{aligned}\lim_{\varepsilon \to 0+} A(\varepsilon, T)f(x) &= \lim_{\varepsilon \to 0+} A(\varepsilon, S)f(x) \\ &= \lim_{\varepsilon \to 0+} [A(\varepsilon, P)(f/h)(x)] \cdot h(x)\end{aligned}$$

$= f(x)$ a.e.

Now consider the case $p = 1$. Assume momentarily that $\| T_t \|_1 \leq M$, $\| T_t \|_\infty \leq M$, $t \geq 0$, and $\mu(X) < \infty$. It is easy to show that Γ is a strongly continuous semigroup of positive $L_p(\mu)$ operators for any given $1 < p < \infty$ (see [4, p. 689]). Since $1 \in L_q(\mu)$ we may set $g = \int_0^\infty S_t^*(1)\, dt$. One sees that $g \in L_\infty(\mu)$. We set $h = (g)^{1/(p-1)}$ and define m and P as before. For $f \in L_1(\mu)$, $f/h \in L_1(m)$ since

$$\int | f/h | h^p d\mu \leq \| g \|_\infty \cdot \int | f | \, d\mu \, .$$

Since the L.E.T. holds for $\{P_t\}$ we have

$$\lim_{\varepsilon \to 0+} A(\varepsilon, T) f(x) = (f/h) h(x) = f(x) \quad \text{a.e.}$$

for $f \in L_1(\mu)$.

In the general case when $p = 1$, we pick $0 < g \in L_1(\mu)$ and define $h = \int_0^\infty S_t g \, dt$. Setting $m(a) = \int_A h \, d\mu$,

$P_t f = S_t(fh)/h$, $f \in L_1(m)$, we have

$\| P_t \|_1 \leq M$, $\| P_t \|_\infty \leq 1$, $t \geq 0$, and $m(X) < \infty$. The special case considered in the preceding paragraph can now be applied to $\{P_t\}$ and we have

$$\lim_{\varepsilon \to 0+} A(\varepsilon, T) f(x) = f(x) \quad \text{a.e.}$$

for $f \in L_1(\mu)$. Q.E.D.

<u>The N-parameter case</u>. The following theorem is an extension of Terrell's result. In order to simplify the notation we consider the case where Γ is a semigroup depending on two parameters. The extension to the general case is imme-

diate. For $f \in L_p(\mu)$ and Γ a 2-parameter semigroup of $L_p(\mu)$ operators, we set

$$A(\varepsilon, T)f(x) = (1/\varepsilon^2) \int_0^{\varepsilon}\int_0^{\varepsilon} T(s,t)f(x)\, dsdt ,$$

where $T(s,t)f(x)$ is a scalar representation of $T(s,t)f$. The definition of $A(\varepsilon, T)f(x)$ in the n-parameter case is completely analogous. As in the one-parameter case there exist $M>0$, $a \geq 0$ such that $\| T(s,t) \|_p \leq Me^{a(s+t)}$, $s, t \geq 0$. For fixed $b > a$, we set $S(s,t) = e^{-b(s+t)}T(s,t)$.

5. Theorem. Let Γ be an n-parameter semigroup of positive $L_p(\mu)$ operators for some $1 \leq p < \infty$. Then the L.E.T. holds for Γ, i.e. $A(\varepsilon, T)f(x) \longrightarrow f(x)$ a.e. as $\varepsilon \longrightarrow 0+$ for $f \in L_p(\mu)$.

Proof. We first establish Terrell's result. For $0 < g \in L_1(\mu)$ set $h = \int_0^{\infty}\int_0^{\infty} S(s,t)g(x) ds\, dt$. Then $0 < h \in L_1(\mu)$ and $S(s,t)h \leq h$ for all $s, t \geq 0$. Defining $m(A) = \int_A h d\mu$ and $P(s,t)f = S(s,t)(fh)/h$ for $f \in L_1(m)$, we have $\| P(s,t) \|_1 \leq 1$, $\| P(s,t) \|_{\infty} \leq 1$, $s, t \geq 0$. By [4, p.697] we have for $\beta > 0$,

$$m\{f^* > \beta\} \leq (K_2/\beta) \int | f | dm ,$$

where $f^* = \sup_{\varepsilon > 0} | A(\varepsilon, P)f(x) |$ and K_2 is independent of f and $\{P(s,t)\}$. Thus $f^* < \infty$ a.e. Since the class $M = \{A(\varepsilon, P)f\colon 0 < \varepsilon < 1, f \in L_1(m)\}$ is dense in $L_1(m)$ and the L.E.T. holds for $f \in M$ (see [12] for a proof), we may apply Banach's convergence principle again to obtain

$$\lim_{\varepsilon \to 0+} A(\varepsilon, P)f(x) = f(x) \quad \text{a.e.}$$

for $f \in L_1(m)$. Since lemma 1 clearly extends to the n-parameter case, we have

$$\lim_{\varepsilon \to 0+} A(\varepsilon, T)f(x) = \lim_{\varepsilon \to 0+} A(\varepsilon, S)f(x)$$
$$= (f/h)h(x)$$
$$= f(x) \quad \text{a.e.}$$

for $f \in L_1(\mu)$. This proves the theorem for the case $p = 1$ and $\| T(s,t) \| \leq 1$.

We now consider the case where $T(s,t)$ is not necessarily a contraction. We assume $p > 1$. Set $g =$
$g = \int_0^\infty \int_0^\infty S^*(s,t)g' \, dsdt$ for some $0 < g' \in L_q(\mu)$. Then

$$S^*(x,t)g = \int_s^\infty \int_t^\infty S^*(u,v)g' \, dudv \leq g$$

for any $s, t \geq 0$. We set $h = g^{1/(p-1)}$, $m(A) = \int_A h^p \, d\mu$, and $P(s,t)f = [S(s,t)(fh)]/h$ for $f \in L_p(m)$. As in the 1-parameter case, we have $\| P(s,t)f \|_1 \leq \| f \|_1$ for any $f \in L_p(m)$ from which it follows that $P(s,t)$ can be extended to a positive $L_1(m)$ contraction. It is easy to show that $\{P(s,t)\}$, regarded as a $L_1(m)$ semigroup, is strongly continuous. For $f \in L_p(\mu)$, $f/h \in L_p(m)$. So

$$\lim_{\varepsilon \to 0+} A(\varepsilon, T)f(x) = (f/h)h(x) = f(x) \quad \text{a.e.}$$

The case $p = 1$ may be handled as in the proof of theorem 4. This concludes the proof.

<u>A conjecture</u>. As a final remark we make the following conjecture concerning an extension of theorem 5. We state our conjecture for the case $n = 2$ to simplify the notation. For ε , $\delta > 0$ and $f \in L_p(\mu)$, set

$$A(\varepsilon,\sigma,T)f(x) = (1/\varepsilon\sigma)\int_0^{\varepsilon}\int_0^{\sigma} T(s,t)f(x)\,ds\,dt ,$$

where $\{T(s,t)\}$ is a semigroup of positive $L_p(\mu)$ operators. We conjecture that if $1 < p < \infty$, then $A(\varepsilon,\sigma,T)f(x) \longrightarrow f(x)$ a.e. as $\varepsilon, \sigma \longrightarrow 0+$ independently. If $M = 1$, i.e. $\| T(s,t) \|_p \leq e^{a(s+t)}$ then the conjecture is true since then $\| S(s,t) \|_p \leq 1$, for $s, t \geq 0$ and consequently $\| f^* \|_p \leq (p/(p-1)) \| f \|_p$, where $f^* = \sup_{\varepsilon,\sigma>0} | A(\varepsilon,\sigma,S)f(x) |$. This estimate for f^* can be obtained using a result in [1]. Upon applying Banach's theorem to $A(\varepsilon,\sigma,S)f(x)$, we get

$$\lim_{\varepsilon,\sigma\to 0+} A(\varepsilon,\sigma,T)f(x) = \lim_{\varepsilon,\sigma\to 0+} A(\varepsilon,\sigma,S)f(x) = f(x) .$$

It is well known (see [3,12]) that the conjecture is false if $p = 1$. Thus it does not appear that the problem may be resolved by using the main technique of this paper, i.e. by introducing the semigroup $\{P(s,t)\}$ of $L_1(m)$ contractions.

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