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ON THE STRUCTURE OF FIXED POINT SETS OF PSEUDO-CONTRACTIVE MAPPINGS

Rainald SCHÖNEBERG, Aachen

Abstract: Let $(E, \| \|)$ be a Banach-space, X a closed and bounded subset of E and let $f: X \longrightarrow E$ be a pseudo-contractive mapping. It is shown that under certain conditions the set Fix(f) of fixed points of f is metrically convex and hence pathwise connected.

Key words: Inward, nonexpansive, pseudo-contractive, k-set-contraction, metrically convex, pathwise connected.

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The purpose of this note is to give some conditions which assure that the fixed point set of a pseudo-contractive mapping is metrically convex and hence pathwise connected. A recent result of the author is basic for the proofs.

Definition 1. Let $(E, \| \|)$ be a Banach-space and $X \subset E$. X is said to be metrically convex: $\Longleftrightarrow$

$$:\Longleftrightarrow \forall_{\substack{x,y \in X \\ x \neq y}} \; \exists_{z \in X} \; z \neq x \wedge z \neq y \wedge \|x - y\| = \|x - z\| + \|y - z\|$$

Remark 1. Every convex set is metrically convex but the converse isn't true in general ($E := \mathbb{R}^2$, $\| \| :=$ max-norm, $X := \{(|t|, t) \mid t \in [-1,1]\}$).

A fundamental property of a metrically convex set is described by

Proposition 1 (K. Menger). Let $(E, \| \ \|)$ be a Banach-space, $X \subset E$ be closed and metrically convex, $x, y \in X$ and $d := \|x - y\|$.

Then there is $\varphi : [o, d] \longrightarrow X$ such that

(i) $\varphi(o) = x \wedge \varphi(d) = y$

(ii) $\forall_{a, b \in [o,d]} \|\varphi(a) - \varphi(b)\| = |a - b|$

(i.e. φ is an isometry)

Proof see [1], Theorem 14.1.

Corollary 1. Let $(E, \| \ \|)$ be a Banach-space and let $X \subset E$ be a closed and metrically convex subset of E.

Then X is pathwise connected.

Proof: Obvious.

Corollary 2. Let $(E, \| \ \|)$ be a strictly convex Banach-space and let $X \subset E$ be closed.

Then X is convex if and only if X is metrically convex.

Proof. If X is convex then X is obviously metrically convex. Conversely suppose X is metrically convex and let $x, y \in X$. By Proposition 1 there is an isometry $\varphi : [o, \|x - y\|] \longrightarrow X$ such that $\varphi(o) = x$ and $\varphi(\|x - y\|) = y$. Since $(E, \| \ \|)$ is strictly convex, φ is affine (see [9]) and hence $\varphi[[o, \|x - y\|]]$ is convex. Therefore $co(\{x,y\}) :=$ convex hull of $\{x,y\} \subset$ $\subset \varphi[[o, \|x - y\|]] \subset X$ i.e. X is convex.

Definition 2. Let $(E, \| \ \|)$ be a Banach-space, $X \subset E$ and let $f: X \longrightarrow E$.

(1) f is said to be nonexpansive: $\Longleftrightarrow \forall_{x,y \in X} \|f(x) - f(y)\| \leq \|x - y\|$

(2) f is said to be pseudo-contractive $:\Longleftrightarrow$

$$:\Longleftrightarrow \underset{x,y\in X}{\forall} \ \underset{r\geq 0}{\forall} \ \| x - y \| \leq \| (1 + r)(x - y) - r(f(x) - f(y)) \|$$

Remark 2. Pseudo-contractive mappings are characterized by the property: f is pseudo-contractive if and only if $Id - f$ is accretive (see [2]). It is easily seen that these mappings include the non-expansive mappings.

In [11] we proved the following theorem:

Theorem. Let $(E, \| \ \|)$ be a Banach-space and suppose M is a closed subset of E such that every nonempty, closed, bounded and convex subset of M possesses the fixed point property with respect to nonexpansive selfmappings. Let $g: M \longrightarrow E$ be nonexpansive such that at least one of the following conditions holds:

(A) M is convex and $g[M] \subset M$

(B) $\mathrm{Fix}\,(g) \cap \partial M = \emptyset$ 1)

Then the (possibly empty) fixed point set of g is metrically convex and hence pathwise connected.

The approach of [4], showing how fixed point theorems for pseudo-contractive mappings may be derived from the fixed point theory of nonexpansive mappings, may be modified to obtain the following two theorems:

Theorem 1. Let $(E, \| \ \|)$ be a Banach-space and suppose X is a nonempty, closed, bounded and convex subset of E such that every nonempty, closed, bounded and convex subset of X possesses the fixed point property with respect to nonexpan-

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1) $\partial M :=$ boundary of M

sive selfmappings. Let $f: X \longrightarrow E$ be a k-set-contraction (in the sense of the Kuratowski-measure of noncompactness [6], $k \geq 0$), pseudo-contractive and inward (i.e.

$$\underset{x \in \partial X}{\forall} \ \underset{u \in X}{\exists} \ \underset{c \geq 0}{\exists} \qquad f(x) = x + c(u - x), \text{ see [3]}).$$

Then Fix(f) is nonempty, bounded, closed and metrically convex.

Proof. Let $\lambda \in (0,1)$ such that $\lambda \cdot k < 1$ and define $T: X \longrightarrow E$ by $T(x) := x - \lambda \cdot f(x)$. Because f is pseudo-contractive we have

(i) $$\underset{x,y \in X}{\forall} \ \| T(x) - T(y) \| \geq (1 - \lambda) \| x - y \|$$

Let now $y \in X$. Defining $h_y: X \longrightarrow E$ by $h_y(x) := \lambda f(x) + (1 - \lambda)y$ it is easily verified that h_y is condensing (because $\lambda \cdot k < 1$) and inward (because f is inward). Hence by [8] there is $x \in X$ with $h_y(x) = x$ i.e. $T(x) = (1 - \lambda)y$. Thus we have shown:

(ii) $$M := (1 - \lambda)X \subset T[X]$$

Because of (i) and (ii) we may define $g: M \longrightarrow M$ by $g(x) := (1 - \lambda)T^{-1}(x)$. Then g is nonexpansive (because of (i)) and every nonempty, closed, bounded and convex subset of M possesses the fixed point property with respect to nonexpansive selfmappings. Since $\mathrm{Fix}(g) = (1 - \lambda)\mathrm{Fix}(f)$ the theorem stated above gives the assertion.

Corollary 3. Let $(E, \| \ \|)$ be a Banach-space, $\emptyset \neq X \subset E$ be closed, bounded an convex and let $f: X \longrightarrow X$ be a k-set-contraction for some $k < 1$ and pseudo-contractive.

Then Fix(f) is nonempty, compact and pathwise connected.

Proof. Let $C_1 := \overline{\mathrm{co}}(f[X])$ and $C_{n+1} := \overline{\mathrm{co}}\,(f[C_n])$ for

$n \geq 1$. Then $C_\infty := \bigcap_{n \geq 1} C_n$ is nonempty, compact and convex such that $f[C_\infty] \subset C_\infty$ (see e.g. [6]). Furthermore $\mathrm{Fix}(f) \subset C_\infty$. Setting $g := f|_{C_\infty}$ Theorem 1 and Corollary 1 yield - observing Schauder's fixed point theorem - that $\mathrm{Fix}(g)$ is nonempty, compact and pathwise connected. Because of $\mathrm{Fix}(f) = \mathrm{Fix}(g)$ we are done.

Theorem 2. Let $(E, \| \ \|)$ be a Banach-space such that every nonempty, closed, bounded and convex subset of E possesses the fixed point property with respect to nonexpansive selfmappings. Let furthermore $X \subset E$ be open and bounded and let $f: \overline{X} \longrightarrow E$ be a k-set-contraction ($k \geq 0$) and pseudo-contractive such that $\mathrm{Fix}(f) \cap \partial X = \emptyset$.
Then the (possibly empty) fixed point set of f is closed, bounded and metrically convex.

Proof. Choose $\lambda \in (0,1)$ such that $\lambda \cdot k < 1$ and define $T: \overline{X} \longrightarrow E$ by $T(x) := x - \lambda f(x)$. Set $M := T[\overline{X}]$. Then M is closed because X is bounded and λf is condensing. Since f is pseudo-contractive we may define $g: M \longrightarrow E$ by $g(x) :=$ $:= (1 - \lambda) T^{-1}(x)$. Then g is nonexpansive. Now Nussbaum's invariance of domain theorem [6] yields that T maps X into the interior of M. Therefore $\partial M \subset T[\partial X]$ which implies that $\mathrm{Fix}(g) \cap \partial M = \emptyset$. Observing $\mathrm{Fix}(f) = (1 - \lambda)\, \mathrm{Fix}(f)$ we are done.

Corollary 4. Let $(E, \langle \, , \, \rangle)$ be a Hilbert-space and let $X \subset E$ be an open, bounded neighborhood of the origin. Let $f: \overline{X} \longrightarrow E$ be a k-set-contraction for some $k \geq 0$ and pseudo-contractive such that

$$\underset{x \in \partial X}{\forall} \ \underset{\lambda \geq 0}{\forall} \quad f(x) = \lambda x \Longrightarrow \lambda < 1.$$

Then Fix(f) is nonempty, closed, bounded and convex.

Proof. Theorem 4 of [10], Theorem 2 and Corollary 2.

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Lehrstuhl C für Mathematik

RWTH Aachen

Templergraben 55, D-5100 Aachen

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