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ON CONVERGENCE OF THE FOURIER SERIES OF A CONSTRUCTIVE FUNCTION OF WEAKLY BOUNDED VARIATION

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Abstract: The paper contains the proofs of some criteria for the convergence of the Fourier series of a constructive function. In particular, the theorem about the convergence of the Fourier series of a constructive function of weakly bounded variation is proved.

Key words: Constructive function, Fourier series of a constructive function, constructive function of weakly bounded variation.

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Introduction. One of the main results of my thesis [9] is proved in the paper. On the basis of the results of the work [9] all the basic theorems of the theory of trigonometric Fourier series hold in constructive analysis. It is possible to prove most of these results by means of the methods closely related to the methods by which they are proved in classical mathematics. This, however, does not hold for the theorem on the convergence of the Fourier series of a constructive function of weakly bounded variation. In classical mathematics this theorem is proved by means of the theorem on the representation of a function of bounded variation as the difference of two non-decreasirg functions. In the constructive analysis this does not hold [7] for constructive functions of weakly bounded variation and therefore it was necessary to find a different method of the proof.

Theorems 1 - 4 are constructive analogues of well-known theorems of the classical theory of Fourier series (see e.g. [8]).

<u>Fundamental definitions</u>. Normal algorithms [3] will be called simply "algorithms". If the algorithm $\mathfrak{A}$ is applicable to the word P, this fact is denoted by $!\,\mathfrak{A}(P)$ and the result of the application of the algorithm $\mathfrak{A}$ to the word P is denoted by $\mathfrak{A}(P)$. The Markov principle is used: For every algorithm $\mathfrak{A}$ and every word P, $\neg\neg\, !\,\mathfrak{A}(P) \supset\, !\,\mathfrak{A}(P)$ always holds. All assertions are to be understood in accordance with the rules of the constructive interpretation of propositions [4].

Natural and rational numbers as well as FR-numbers are defined in [5]. We shall often use the term "point" instead of "FR-number". An algorithm f is called a <u>constructive function</u> of a real variable (or only function) [6] if it satisfies the following two conditions:

1) For any FR-number x, if $!f(x)$, then $f(x)$ is FR-number;

2) $\forall xy\, (!f(x) \,\&\, x = y \supset\, !f(y) \,\&\, f(x) = f(y))$.

Any constructive function is continuous in any point in which it is defined [6]. We shall use (with or without strokes and subscripts)

m, n as variables which vary through natural numbers,

j,k,ℓ as variables which vary through integers, $a,b,c,t,u,v,x,y,\alpha,\beta,\gamma,\delta,\mu,\xi$ as variables which vary through FR-numbers and f,g,h as variables which vary through functions.

A set will be understood as a set of FR-numbers, i.e. the word M of the form $\wedge p\,\mathcal{F}(p)$, where p is one of the variables for FR-numbers and $\mathcal{F}(p)$ is a one-parameter formula [4] with the parameter p. For this set M we define: $q \in M \leftrightharpoons \mathcal{F}(q)$, where q is also one of the variables for FR-numbers. We denote by R the set of all FR-numbers. Let M_1 and M_2 be two sets. We define: $M_1 \subseteq M_2 \leftrightharpoons \forall x\,(x \in M_1 \supset x \in M_2)$. The set $\wedge x\,(a \leqslant x \leqslant b)$ (where a and b are expressions denoting FR-numbers and $a \leqslant b$) is called a segment; we denote this set by $a \Delta b$. We denote: $a \nabla b \leftrightharpoons \wedge x\,(a < x < b)$.

A function f defined on the segment $a \Delta b$ is called a <u>function of weakly bounded variation</u> on $a \Delta b$ if there exists a FR-number u such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leqslant u \text{ for } a \leqslant x_0 \leqslant x_1 \leqslant \dots \dots \leqslant x_n \leqslant b.$$

We understand the notions of sequence of FR-numbers and functions, their convergence, polygonal function on a segment and uniformly continuous function on a segment in the usual (constructive) sense. We denote:

$$x_n \xrightarrow[m\to\infty]{} x \leftrightharpoons \forall m\, \exists n_0\, \forall n\,(n \geq n_0 \supset |x_n - x| < 2^{-m}).$$

The concept of Lebesgue integrability and Lebesgue integral of a constructive function is defined in [1]. We denote $\mathcal{L}(f, a\Delta b)$ if f is defined on the segment $a \Delta b$ and

f is Lebesgue integrable in $a \Delta b$. One can construct an algorithm which for every a and b (where $a< b$), every function f such that $\mathcal{L}(f,a\Delta b)$ produces on the basis of the sufficient information about the function f (this information will be denoted by $[f]$) and on the basis $u,v \in a\Delta b$ the value of the Lebesgue integral of the function f from u to v. The value of the Lebesgue integral of the function f from u to v will be denoted by $\int_u^v [f(x)]\, dx$.

Let us note that if f is of weakly bounded variation on $a\Delta b$, then f is Lebesgue integrable on $a\Delta b$ (see [1], Theorem 1).

The following lemmas can be proved easily immediately from the definition of the Lebesgue integral. (In Lemma 1 we denote the composition of the functions f,g by $f \circ g$ and the inverse function of the function g by g_{-1}.)

<u>Lemma 1</u>. Let c and d be FR-numbers, let $c \neq 0$ and let g be such that $\forall x\, (g(x) = c \cdot x + d)$. Then

$$(*)\quad \mathcal{L}(f,a\Delta b) \equiv \mathcal{L}(f \circ g,\ \min(g_{-1}(a),g_{-1}(b))\Delta \max(g_{-1}(a),g_{-1}(b)))$$

and $\int_a^b [f(x)]\, dx = c \cdot \int_{g_{-1}(a)}^{g_{-1}(b)} [f(c\cdot x + d)]\, dx$

if one side in $(*)$ holds.

<u>Lemma 2</u>. Let $\mathcal{L}(f,a\Delta b)$, let f be periodic with period u and let $u < b - a$. Then for every α and β (where $\alpha < \beta$), $\mathcal{L}(f, \alpha\Delta\beta)$ and for every FR-number c,

$$\int_a^{a+u} [f(x)]\, dx = \int_c^{c+u} [f(x)]\, dx.$$

The functions sin and cos and FR-number π can be defined by means of the well-known power series so that they have well-known properties especially those we shall use.

Fourier series

Definition. Let $\mathcal{L}(f,0\vartriangle 2\pi)$. Then obviously there exist the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ such that

$$(1)\quad \forall k\ ((k\geq 0 \supset a_k = \frac{1}{\pi}\cdot\int_0^{2\pi}[f(x)\cdot\cos kx]\,dx)\ \&$$

$$\&(k>0 \supset b_k = \frac{1}{\pi}\cdot\int_0^{2\pi}[f(x)\cdot\sin kx]\,dx)).$$

These sequences will be called the sequences of the Fourier coefficients of the function f; a_k, b_k will be called the Fourier coefficients of the function f;

we shall denote by s_m^f the functions such that

$$(2)\quad \forall x(s_m^f(x) = \frac{1}{2}\cdot a_0 + \sum_{k=1}^{m}(a_k\cdot\cos kx + b_k\cdot\sin kx)).$$

The sequence $\{s_m^f\}_m$ will be called the Fourier series of the function f.

Notation. We shall denote by S_n the functions such that

$$(3)\qquad \forall x(S_n(x) = 1 + 2\cdot\sum_{k=1}^{n}\cos 2kx).$$

If $\mathcal{L}(f,0\vartriangle 2\pi)$ and f is periodic with period 2π, we shall denote this fact by $\mathcal{P}(f,0\vartriangle 2\pi)$.

Theorem 1 can be proved easily in the same way as in classical mathematics (see e.g. [8]).

Theorem 1. Let $\mathcal{L}(f,0\vartriangle 2\pi)$. Then $\forall mx\ (s_m^f(x) =$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} [f(u) \cdot S_m(\frac{u-x}{2})] du) .$$

If $\mathcal{P}(f, 0 \Delta 2\pi)$, then

$$(4) \quad \forall mx(s_m^f(x) = \frac{1}{2\pi} \cdot \int_{x-\pi}^{x+\pi} [f(u) \cdot S_m(\frac{u-x}{2})] \, du \; \&$$

$$\& \; s_m^f(x) = \frac{1}{\pi} \cdot \int_0^{\frac{\pi}{2}} [(f(x + 2t) + f(x - 2t)) \cdot S_m(t)] \, dt).$$

In particular, if f is a function such that $\forall x(f(x) = 1)$, we obtain

$$\forall m(1 = \frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} [S_m(t)] \, dt \; \& \; \forall x(1 = \frac{1}{2\pi} \cdot \int_0^{2\pi} [S_m(\frac{u-x}{2})] \, du \; \&$$

$$\& \; 1 = \frac{1}{2\pi} \cdot \int_{x-\pi}^{x+\pi} [S_m(\frac{u-x}{2}] \; du)).$$

Theorem 2. Let f be of weakly bounded variation on $a \Delta b$. Then there exists a FR-number A such that

$$(5) \quad \forall \mu \alpha \beta (\mu > 0 \; \& \; a \prec \alpha < \beta \prec b \prec f \Rightarrow$$

$$\Rightarrow |\int_\alpha^\beta [f(x) \cdot \cos \mu x] dx| \prec \frac{A}{\mu} \; \& \; |\int_\alpha^\beta [f(x) \cdot \sin \mu x] dx| \prec \frac{A}{\mu}) .$$

Proof. There exists u such that for every non-decreasing finite sequence $\{\alpha_k\}_{k=0}^m$ of FR-numbers in $a \Delta b$ we have

$$(6) \qquad \sum_{k=1}^{m} | f(\alpha_k) - f(\alpha_{k-1}) | \prec u.$$

Then obviously also there exists v such that

$$(7) \qquad \forall x(x \in a \Delta b \Rightarrow | f(x) | \prec v).$$

Let $A \leftrightharpoons 2 \cdot (u + 3 \cdot v)$, $\mu > 0$ and $a \prec \alpha < \beta \prec b$. According to Lemma 1

$$(8) \qquad \int_{\alpha}^{\beta} [f(x) \cdot \sin \mu x] dx = \frac{1}{\mu} \cdot \int_{\mu\alpha}^{\mu\beta} [f(\tfrac{y}{\mu}) \cdot \sin y] dy .$$

Obviously $\neg\neg \exists k \ell ((k-1) \cdot \pi \leqq \mu \cdot \alpha < k \cdot \pi \,\&\, \ell \cdot \pi \leqq \mu \cdot \beta < (\ell+1) \cdot \pi)$.

Let

$$(9) \qquad (k-1) \cdot \pi \leqq \mu \cdot \alpha < k \cdot \pi \,\&\, \ell \cdot \pi \leqq \mu \cdot \beta < (\ell+1) \cdot \pi .$$

Then obviously $k - 1 \leqq \ell$. One can construct the non-decreasing finite sequence $\{ y_j \}_{j=k-1}^{\ell+1}$ such that

$$y_{k-1} = \mu \cdot \alpha \,\&\, \forall j \, (k \leqq j \leqq \ell \supset y_j = j \cdot \pi) \,\&\, y_{\ell+1} = \mu \cdot \beta .$$

We have

$$(10) \qquad \int_{\mu\alpha}^{\mu\beta} [f(\tfrac{y}{\mu}) \cdot \sin] dy = \sum_{j=k}^{\ell+1} \int_{y_{j-1}}^{y_j} [f(\tfrac{y}{\mu}) \cdot \sin y] dy .$$

Let $k \leqq j \leqq \ell + 1$. Then $y_{j-1} \triangle y_j \subseteq (j-1) \cdot \pi \triangle j \cdot \pi$.

We have $\forall y \, (y \in (j-1) \cdot \pi \triangle j \cdot \pi \supset \sin y = (-1)^{j-1} \cdot |\sin y|)$.

Hence

$$(11) \qquad \int_{y_{j-1}}^{y_j} [f(\tfrac{y}{\mu}) \cdot \sin y] dy = (-1)^{j-1} \int_{y_{j-1}}^{y_j} [f(\tfrac{y}{\mu}) \cdot |\sin y|] dy .$$

According to Lemma 1

$$(12) \qquad \int_{y_{j-1}}^{y_j} [f(\tfrac{y}{\mu}) \cdot |\sin y|] dy = \int_{y_{j-1}-(j-1)\pi}^{y_j-(j-1)\pi} [f(\tfrac{t}{\mu} + \tfrac{j-1}{\mu} \pi) \cdot |\sin t|] dt.$$

Let us suppose that $k < \ell + 1$. From (10) - (12) we obtain

$$(13) \qquad \begin{cases} \int_{\mu\alpha}^{\mu\beta} [f(\tfrac{y}{\mu}) \cdot \sin y] dy = (-1)^{k-1} \cdot \int_{\mu\alpha-(k-1)\cdot\pi}^{\pi} [f(\tfrac{y}{\mu} + \tfrac{k-1}{\mu} \pi) \cdot \\ \cdot |\sin y|] dy + \sum_{j=k+1}^{\ell} (-1)^{j-1} \cdot \int_0^{\pi} [f(\tfrac{y}{\mu} + \tfrac{j-1}{\mu} \pi) \cdot \\ \cdot |\sin y|] dy + (-1)^{\ell} \cdot \int_0^{\mu\beta-\ell\pi} [f(\tfrac{y}{\mu} + \tfrac{\ell}{\mu} \pi) \cdot |\sin y|] dy . \end{cases}$$

Because of (13), (8), (7) and (9)

(14) $$\left|\int_{\alpha}^{\beta}[f(x)\cdot \sin \mu x]\,dx\right| \leq \frac{1}{\mu}\cdot \Big(2\cdot v\cdot \int_0^{\pi}[\sin y]\,dy + \int_0^{\pi}\Big[\Big|\sum_{j=k+1}^{\ell}(-1)^{j-1}\cdot f\Big(\frac{y}{\mu}+\frac{j-1}{\mu}\pi\Big)\Big|\cdot \sin y\Big]\,dy\Big).$$

Let k = ℓ + 1. By (9) then $\ell\cdot\pi \leq \mu\cdot\alpha < \mu\cdot\beta < (\ell+1)\cdot\pi$, hence by (8) and (7)

$$\left|\int_{\alpha}^{\beta}[f(x)\cdot \sin \mu x]\,dx\right| \leq \frac{1}{\mu}\cdot v\cdot \int_{\mu\alpha}^{\mu\beta}[|\sin y|]\,dy \leq \frac{v}{\mu}\cdot \int_{\ell\cdot\pi}^{(\ell+1)\pi}[|\sin y|]\,dy,$$

hence (14) holds also for k = ℓ + 1. In view of (6),(7) and (9)

(15) $$\forall y\Big(y\in 0\,\Delta\,\pi \supset \Big|\sum_{j=k+1}^{\ell}(-1)^{j-1}\cdot f\Big(\frac{y}{\mu}+\frac{j-1}{\mu}\pi\Big)\Big| \leq u+v\Big).$$

From (14) and (15) it follows

(16) $$\left|\int_{\alpha}^{\beta}[f(x)\cdot \sin \mu x]\,dx\right| \leq \frac{1}{\mu}\cdot (u+3\cdot v)\cdot \int_0^{\pi}[\sin y]\,dy = \frac{A}{\mu}.$$

Let us denote the formulas (9) and (16) by $\mathcal{A}$ and $\mathcal{B}$ respectively. We have proved $\mathcal{A} \supset \mathcal{B}$, hence $\exists k\ell\, \mathcal{A} \supset \mathcal{B}$, hence $\neg\neg\exists k\ell\,\mathcal{A} \supset \neg\neg\mathcal{B}$ and because $\neg\neg\exists k\ell\,\mathcal{A}$, we have $\neg\neg\mathcal{B}$, hence (16). The assertion (5) has been proved for sin, for cos it can be proved analogously.

<u>Theorem 3</u>. Let $\mathcal{L}(g, a\Delta b)$. Then $\int_{\alpha}^{\beta}[g(x)\cdot \sin\mu x]dx \xrightarrow[\mu\to+\infty]{} 0$ uniformly with respect to $\alpha, \beta \in a\,\Delta\, b$ and $\int_{\alpha}^{\beta}[g(x)\cdot \cos\mu x]\,dx \xrightarrow[\mu\to+\infty]{} 0$ uniformly with respect to $\alpha,\beta \in a\,\Delta\, b$, i.e.

(17) $$\forall m\,\exists\mu_0\,\forall\mu\alpha\beta\Big(\mu>\mu_0\,\&\,\alpha,\beta\in a\,\Delta\, b \supset \left|\int_{\alpha}^{\beta}[g(x)\cdot\cos\mu x]\,dx\right| < 2^{-m}\,\&\,\left|\int_{\alpha}^{\beta}[g(x)\cdot\sin\mu x]\,dx\right| < 2^{-m}\Big).$$

In particular, if $\mathcal{L}(f, 0 \triangle 2\pi)$ and $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ are the sequences of the Fourier coefficients of the function f, then $a_k \xrightarrow[k\to\infty]{} 0$ and $b_k \xrightarrow[k\to\infty]{} 0$.

Proof. By Theorem 4 in [1] it follows from the definition of Lebesgue integral that for n there exists a function f of bounded variation on $a \triangle b$ (even polygonal on $a \triangle b$) such that $\int_a^b [\,|g(x) - f(x)|\,]\,dx < 2^{-n-1}$. By Theorem 2 we have (5). Let $\mu_0 \gneqq 2^{n+1} \cdot A$. Then for $\mu > \mu_0$ (hence $2^{-n-1} > \frac{A}{\mu}$) and $\alpha, \beta \in a \triangle b$,

$$|\int_\alpha^\beta [g(x) \cdot \cos \mu x]\,dx| \leq \int_a^b [\,|g(x) - f(x)|\,]\,dx \; +$$

$$+ \,|\int_\alpha^\beta [f(x) \cdot \cos \mu x]\,dx\,| < 2^{-n-1} + \frac{A}{\mu} < 2^{-n} \; .$$

The assertion (17) has been proved for cos, for sin it can be proved analogously.

Lemma 3 is proved in the same way as in classical mathematics (see e.g. [8], the proof of Theorem 31) on the basis of Theorem 3 and using the second mean-value theorem for Lebesgue integral (see [2], p. 264).

Lemma 3. Let $\mathcal{P}(f, 0 \triangle 2\pi)$, let $0 < \delta \leq c \leq \frac{\pi}{2}$ and let $\{g_n\}_n$ be the sequence of functions such that

$$\forall n x \,(g_n(x) = \int_\delta^c [(f(x+2t) + f(x-2t)) \cdot S_n(t)]\,dt) \; .$$

Then $\{g_n\}_n$ converges uniformly on R to zero.

Notation. If $\mathcal{P}(f, 0 \triangle 2\pi)$, then we shall denote for a fixed δ by $s_n^{f\delta}$ the function such that

$$\forall x\,(s_n^{f\delta}(x) = \frac{1}{\pi} \cdot \int_0^\delta [(f(x+2t) + f(x-2t)) \cdot S_n(t)]\,dt) \; .$$

Theorem 4. Let $\mathcal{P}(f, 0 \triangle 2\pi)$ and let $0 < \delta < \frac{\pi}{2}$.
Then $\{s_m^f - s_m^{f\delta}\}_m$ converges uniformly on R to zero.
In particular, if f is such that $\forall x(f(x) = 1)$, then we get $\frac{2}{\pi} \cdot \int_0^{\delta} [S_m(t)]\,dt \xrightarrow[m\to\infty]{} 1$.

Proof. By Theorem 1 we have

$$s_m^f(x) - s_m^{f\delta}(x) = \frac{1}{\pi} \cdot \int_{\delta}^{\frac{\pi}{2}} [(f(x+2t) + f(x-2t)) \cdot S_m(t)]\,dt .$$

Hence by Lemma 3 (where $c = \frac{\pi}{2}$) the assertion of the theorem holds.

Lemma 4. Let for f,m,n and FR-numbers $a, b, A, x, \varepsilon, \delta$ it holds $a < b$, $\mathcal{L}(f, a \triangle b)$, $\varepsilon > 0$, $a + \varepsilon < x < b - \varepsilon$,
$0 < \delta < min\left(\frac{\pi}{2}, \frac{\varepsilon}{2}\right)$, $m > \frac{1}{2} \cdot \left(\frac{\pi \cdot m}{\delta} - 1\right)$ and

$$(18)\ \forall \alpha \beta \left(a < \alpha < \beta < b \Rightarrow \left|\int_{\alpha}^{\beta} [f(u) \cdot sin(2m+1)\frac{u-x}{2}]\,du\right| < \frac{A}{2 \cdot m + 1}\right).$$

Then

$$(19)\ \left|\int_{\frac{\pi m}{2m+1}}^{\delta} [\varphi(x,t) \cdot S_m(t)]\,dt\right| < \frac{A + 4 \cdot |f(x)|}{2 \cdot m}, \quad \text{where}$$

$$(20) \qquad \varphi(x,t) = f(x + 2t) + f(x - 2t) - 2 \cdot f(x).$$

Proof. Let the assumptions of the lemma be satisfied. Then

$\delta > \frac{\pi \cdot m}{2 \cdot m + 1}$. Let $\gamma \leftrightharpoons \frac{\pi \cdot m}{2 \cdot m + 1}$. Then $0 < \gamma < \delta < \frac{\pi}{2}$,

hence

$$\int_{\gamma}^{\delta} [\varphi(x,t) \cdot S_m(t)]\,dt = \int_{\gamma}^{\delta} [\varphi(x,t) \cdot \frac{sin(2m+1)t}{sin\,t}]\,dt = \int_{\gamma}^{\delta} [h(t) \cdot g(t)]\,dt,$$

where h and g are functions such that

(21) $\forall t (h(t) = \varphi(x,t) \cdot \sin(2m+1)t)$

and $\forall t (t \in \gamma \Delta \delta \supset g(t) = \frac{1}{\sin t})$

Obviously $\mathcal{L}(h, \gamma \Delta \delta)$, g is non-increasing on $\gamma \Delta \delta$

and $\frac{1}{\sin \gamma} \geq g(\gamma) \geq g(\delta) \geq 0$, hence by the second mean-value theorem for Lebesgue integral ([2], p.264)

$$(22) \qquad \neg\neg \exists \xi \left(\gamma \leq \xi \leq \delta \;\&\; \int_\gamma^\delta [h(t) \cdot g(t)]\,dt = \frac{1}{\sin \gamma} \cdot \int_\gamma^\xi [h(t)]\,dt\right).$$

By using Lemma 1 we get for $\xi \in \gamma \Delta \delta$:

$$(23) \quad \begin{cases} \int_\gamma^\xi [h(t)]\,dt = \int_\gamma^\xi [\varphi(x,t) \cdot \sin(2m+1)t]\,dt = \\ = \frac{1}{2} \cdot \left(\int_{x+2\gamma}^{x+2\xi} [f(u) \cdot \sin(2m+1)\frac{u-x}{2}]\,du - \right. \\ \left. - \int_{x-2\xi}^{x-2\gamma} [f(u) \cdot \sin(2m+1)\frac{u-x}{2}]\,du\right) + \\ + 2 \cdot f(x) \cdot \frac{\cos(2m+1)\xi - \cos(2m+1)\gamma}{2 \cdot m + 1} \end{cases}.$$

We have $a < x + 2 \cdot \gamma \leq x + 2 \cdot \xi \leq x + 2 \cdot \delta < x + \varepsilon \leq b$ and $a \leq x - \varepsilon < x - 2 \cdot \delta \leq x - 2 \cdot \xi \leq x - 2 \cdot \gamma < b$. Hence, in view of (23) and (18)

$\left|\int_\gamma^\xi [h(t)]\,dt\right| \leq \frac{A + 4 \cdot |f(x)|}{2 \cdot m + 1}$. Because of (22) thus

$$(24) \quad \left|\int_\gamma^\delta [\varphi(x,t) \cdot S_m(t)]\,dt\right| = \left|\int_\gamma^\delta [h(t) \cdot g(t)]\,dt\right| \leq \frac{A + 4 \cdot |f(x)|}{(2 \cdot m + 1) \cdot \sin \gamma}.$$

We have $0 < \gamma < \frac{\pi}{2}$, thus $\sin \gamma \geq \frac{2}{\pi} \cdot \gamma = \frac{2 \cdot m}{2 \cdot m + 1}$. In view of (24) we have (19).

<u>Notation.</u> We denote $M \subseteq_1 a \Delta b$ if $a < b$, M is a set and

there exists a FR-number $\varepsilon > 0$ such that $M \subseteq (a+\varepsilon) \triangle (b-\varepsilon)$.

Theorem 5. Let $\mathcal{S}(f, 0 \triangle 2\pi)$, $M \subset_1 a \triangle b$ and let there exist a FR-number A such that

$$(25)\quad \forall m x \alpha \beta (x \in M \ \& \ a < \alpha < \beta < b \supset$$
$$\supset |\int_\alpha^\beta [f(u) \cdot \sin(2m+1) \frac{u-x}{2}] du| < \frac{A}{2 \cdot m+1}) .$$

Then Fourier series of the function f converges on M to f.

If moreover f is uniformly continuous on a segment containing M, then the Fourier series of the function f converges uniformly on M to f.

Proof. Obviously it suffices to prove only the special assertion. Let us choose a fixed $n > 0$; there exists a FR-number $\varepsilon > 0$ such that $M \subseteq (a+\varepsilon) \triangle (b-\varepsilon)$; f is uniformly on a segment containing M; thus there exists δ such that

$$(26)\qquad 0 < \delta < \min\left(\frac{\pi}{2}, \frac{\varepsilon}{2}\right) \qquad \text{and}$$

$$(27)\quad \forall x t (x \in M \ \& \ t \in 0 \triangle \delta \supset |\varphi(x,t)| < \frac{1}{n^2} ,$$

where $\varphi(x,t)$ is the same as in (20). In view of (26) by Theorem 4 there exists $m_0 > \frac{1}{2} \cdot \left(\frac{\pi \cdot n}{\delta} - 1\right)$ such that

$$(28)\ \forall m x (m \geq m_0 \supset |s_m^f(x) - s_m^{f\delta}(x)| < \frac{1}{n} \ \& \ |\frac{2}{\pi} \cdot \int_0^\delta [S_m(t) dt - 1| < \frac{1}{n}).$$

Let $m \geq m_0$ and $x \in M$. Let us denote $\gamma \Leftrightarrow \frac{\pi \cdot n}{2 \cdot m+1}$. By Lemma 4

$$(29)\quad |\int_\gamma^\delta [\varphi(x,t) \cdot S_m(t)] dt| < \frac{A + 4 \cdot |f(x)|}{2 \cdot n} .$$ We have

(30) $s_m^{f\delta}(x)-f(x)=\frac{1}{\pi}\cdot\int_0^{\delta}[\varphi(x,t)\cdot S_m(t)]dt+f(x)\cdot$

$\cdot(\frac{2}{\pi}\cdot\int_0^{\delta}[S_m(t)]dt-1)$ and

(31) $\int_0^{\delta}[\varphi(x,t)\cdot S_m(t)]dt=\int_0^{\gamma}[\varphi(x,t)\cdot S_m(t)]dt+\int_{\gamma}^{\delta}[\varphi(x,t)\cdot S_m(t)]dt.$

Clearly $0<\gamma<\delta$, thus in view of (27)

$|\int_0^{\gamma}[\varphi(x,t)\cdot S_m(t)]dt|\leq\frac{1}{m^2}\cdot\int_0^{\gamma}[|S_m(t)|]dt$. Obviously for every t we have $|S_m(t)|\leq 1+2\cdot m$, thus

$\int_0^{\gamma}[|S_m(t)|]dt\leq(2\cdot m+1)\cdot\gamma=\pi\cdot m$, hence

(32) $$|\int_0^{\gamma}[\varphi(x,t)\cdot S_m(t)dt|\leq\frac{\pi}{m}\ .$$

In view of (28) - (32) we have

$|s_m^f(x)-f(x)|\leq|s_m^f(x)-s_m^{f\delta}(x)|+$

$+|s_m^{f\delta}(x)-f(x)|\leq\frac{1}{m}\cdot(2+|f(x)|+\frac{A+4\cdot|f(x)|}{2\cdot\pi})$.

Obviously there exists a FR-number B such that

$\forall x(x\in M\Rightarrow|f(x)|\leq B)$, hence $|s_m^f(x)-f(x)|\leq\frac{1}{m}\cdot(2+B+\frac{A+4\cdot B}{2\pi})$.

We have proved that there exists a FR-number $C>0$ such that

$\forall n\,\exists m_0\,\forall m\,x\,(m\geq m_0\ \&\ x\in M\Rightarrow|s_m^f(x)-f(x)|\leq\frac{C}{m})$.

Thus the special assertion holds.

An immediate consequence of this theorem is the following theorem.

Theorem 6. Let $\mathcal{P}(f,0\Delta 2\pi)$ and there exist FR-numbers a,b,K such that a< b and

$$(33)\quad \forall m\,\alpha\,\beta\,(a<\alpha<\beta<b \supset |\int_{\alpha}^{\beta}[f(u)\cdot\sin(m+\frac{1}{2})u]du| + |\int_{\alpha}^{\beta}[f(u)\cdot\cos(m+\frac{1}{2})u]du| < \frac{K}{2\cdot m+1}) .$$

Then (a) the Fourier series of the function f converges on $a \triangledown b$ to f and

(b) for every A and B such that, $a<A<B<b$ and f is uniformly continuous on $A\triangle B$, the Fourier series of the function f converges uniformly on $A\triangle B$ to f.

<u>Theorem 7</u>. Let $\mathcal{P}(f,0\triangle 2\pi)$ and let f be of weakly bounded variation on $a\triangle b$.

Then (a) the Fourier series of the function f converges on $a \triangledown b$ to f and

(b) for every A and B such that $a<A<B<b$ and f is uniformly continuous on $A\triangle B$, the Fourier series of the function f converges uniformly on $A\triangle B$ to f.

<u>Proof</u>. By Theorem 2 there exists a FR-number K such that (33) holds. Hence by Theorem 6 the assertion of the theorem holds.

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