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ON COMPACT SPACES WHICH ARE UNIONS OF CERTAIN COLLECTIONS OF SUBSPACES OF SPECIAL TYPE

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Abstract: Let X be a compact and $X = X_1 \cup X_2$ where both X_1 and X_2 are metrizable. Then X need not be metrizable itself but X must be a Fréchet-Urysohn space and for every $A \subset X$ the cellularity number $c(A)$ is equal to the weight $w(A)$ (Theorem II.10). If X is a compact and $X = \bigcup \mathcal{F}$ where each $Y \in \mathcal{F}$ is a developable subspace of X then the tightness of X is countable (Theorem I.12). Together with these results we prove a few useful general lemmas. The following problem is formulated (see II.13). Let $X = X_1 \cup X_2$ where X is a compact and X_1, X_2 are metrizable. Is it true then that X is an Eberlein compact?

Key words and phrases: Tightness, density, Fréchet-Urysohn space, sequential space, free sequence, π-weight, network, developable space, uniform base.

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0. Conventions and notations. Throughout the paper the word "space" will mean "topological regular T_1-space". "A compact" is a bicompact Hausdorff space. The symbol will always denote a cardinal number. We shall write τ^+

for the first cardinal which is greater than τ . Cardinals are identified with the corresponding initial ordinals. We put $N^+ = \{1,2,\dots,n,\dots\}$. If X is a space, $A \subset X$ and τ is a cardinal, then $c\ell(A)$ is the closure of A in X, $c\ell_\tau(A) = \bigcup\{c\ell(M): M \subset A$ and $|M| \le \tau\}$ and $\operatorname{seqc}\ell(A) = \{x \in X$: there exists a sequence $\{a_n: n \in N^+\}$ in A converging to $x\}$. A transfinite sequence $\xi = \{x_\alpha : \alpha < \tau^+\}$ of points in X is called "a free sequence" (see [3]), if for each $\beta < \tau^+$ the following condition holds:

$c\ell\{x_\alpha: \alpha < \beta\} \cap c\ell\{x_\alpha: \beta \le \alpha < \tau^+\} = \emptyset$. Then τ^+ is called "the length" of ξ and we write: $\ell(\xi) = \tau^+$. The set of all free sequences in X of the length τ^+ will be denoted by $\mathcal{F}_\tau(X)$. A space X is called "a τ-compact" if for every chain C of non-empty closed sets in X such that $|C| \le \tau$ we have: $\bigcap C \neq \emptyset$. We also consider the following cardinal-valued invariants: tightness $t(X)$ of X, density $d(X)$ of X, cellularity number (Souslin number) $c(X)$ of X, pseudocharacter $\psi(X)$ of X, character $\chi(X)$ of X and with some others. Their definitions one can find in [5]. The cardinality of a set X is denoted by $|X|$.

§ I. General results

I.1. Definition. Let $\xi = \{x_\alpha : \alpha < \tau^+\} \in \mathcal{F}_\tau(X)$ and $\xi' = \{x'_\alpha : \alpha < \tau^+\} \in \mathcal{F}_\tau(X)$. We put $\xi' < \xi$ if $(c\ell_\tau\{x_\alpha : \beta \le \alpha < \tau^+\}) \setminus \{x_\alpha : \beta \le \alpha < \tau^+\} \supset \{x'_\alpha : \beta \le \alpha < \tau^+\}$. We shall write $\xi_1 \le \xi_2$, if

$\xi_1 < \xi_2$ or $\xi_1 = \xi_2$. Obviously, if $\xi'' < \xi'$ and $\xi' < \xi$, then $\xi'' < \xi$ - one only has to remark that always $c\ell_\tau(c\ell_\tau(A)) = c\ell_\tau(A)$ (see [3]).

I.2. Lemma. If $\xi, \xi' \in \mathcal{F}_\tau(X)$ and $\xi' < \xi$, then $c\ell_\tau(\xi')$ is closed in $c\ell_\tau(\xi)$ and $c\ell_\tau(\xi') \setminus \xi'$ is closed in $c\ell_\tau(\xi) \setminus \xi$.

Proof. Let $y \in c\ell_\tau(\xi)$ and $y \in c\ell(c\ell_\tau(\xi'))$. Then $y \in c\ell\{x_\alpha : \alpha < \alpha^*\}$ for some $\alpha^* < \tau^+$. Then $y \notin c\ell\{x_\alpha : \alpha^* \le \alpha < \tau^+\}$ and hence $y \notin c\ell\{x'_\alpha : \alpha^* \le \alpha < \tau^+\}$. Thus $y \in c\ell\{x'_\alpha : \alpha < \alpha^*\}$. We conclude that $y \in c\ell_\tau(\xi')$. As ξ and ξ' are discrete subspaces of X, ξ is open in $c\ell_\tau(\xi)$ and ξ' is open in $c\ell_\tau(\xi')$. From this the second conclusion of the lemma follows.

I.3. Lemma. If $\xi \in \mathcal{F}_\tau(X)$ and $M \subset c\ell_\tau(\xi)$, $|M| \le \tau$, $\tau \ge \aleph_0$, then $M \subset c\ell(\{x_\alpha : \alpha < \alpha^*\})$ for some $\alpha^* < \tau^+$.

This assertion follows trivially from regularity of τ^+ .

I.4. Lemma. If X is a τ-compact and $A \subset X$ then $Y = c\ell_\tau(A)$ is also a τ-compact.

Proof. Let C be a chain of non-empty closed sets in Y such that $|C| \le \tau$. For each $F \in C$ let us fix $c(F) \in F$. We put $M = \{c(F) : F \in C\}$. Then $|M| \le |C| \le \tau$ and $M \subset Y$. Hence $c\ell(M) \subset c\ell_\tau(Y) = Y$. As $c(F) \in M \cap F$, the family $C' = \{F \cap c\ell M : F \in C\}$ is a chain of non-empty closed sets in $c\ell(M)$ and, hence, in X. As $|C'| \le |C| \le \tau$ we can conclude that $\bigcap C' \ne \emptyset$. But $\bigcap C' \subset \bigcap C$. Thus $\bigcap C \ne \emptyset$.

I.5. Lemma. Let X be a τ-compact, $\theta^* < \tau^+$, $\tau \geq \aleph_0$ and suppose that for every $\theta < \theta^*$ a free sequence $\xi_\theta = \{x_\alpha^\theta : \alpha < \tau^+\} \in \mathcal{F}_\tau(X)$ is given in such a way that if $\theta' < \theta'' < \theta^*$, then $\xi_{\theta''} < \xi_{\theta'}$. Then there exists a free sequence $\xi_{\theta^*} = \{x_\alpha^{\theta^*} : \alpha < \tau^+\}$ in X such that $\xi_{\theta^*} < \xi_\theta$ for all $\theta < \theta^*$.

Proof. For all $\theta < \theta^*$ and $\alpha < \tau^+$ we put $F_\alpha^\theta = (c\ell_\tau\{x_\beta^\theta : \alpha \leq \beta < \tau^+\}) \setminus \{x_\beta^\theta : \alpha \leq \beta < \tau^+\}$. We shall define a transfinite sequence $\eta = \{y_\alpha : \alpha < \tau^+\}$ in X. Let $\tilde{\alpha} < \tau^+$. Assume that for every $\alpha < \tilde{\alpha}$ a point $y_\alpha \in X$ is already defined in such a way that $y_\alpha \in c\ell_\tau(\xi_0)$. In view of the lemma I.3, there exists $\alpha^* < \tau^+$ such that $\{y_\alpha : \alpha < \tilde{\alpha}\} \subset c\ell\{x_\alpha^0 : \alpha < \alpha^*\}$ and $\tilde{\alpha} \leq \alpha^* < \tau^+$. The smallest α^*, for which the two conditions above are satisfied will be denoted by $\varphi(\tilde{\alpha})$. Clearly $\tilde{\alpha} \leq \varphi(\tilde{\alpha}) < \tau^+$. We put $\tilde{C} = \{F_{\varphi(\tilde{\alpha})}^\theta : \theta < \theta^*\}$ and $\Phi_{\tilde{\alpha}} = \bigcap \tilde{C}$. By I.4, $F_{\varphi(\tilde{\alpha})}^0$ is a τ-compact. By I.2, each $F_{\varphi(\tilde{\alpha})}^\theta$ is closed in $F_{\varphi(\tilde{\alpha})}^0$. If $F_{\varphi(\tilde{\alpha})}^\theta = \emptyset$ then $c\ell_\tau\{x_\beta^\theta : \varphi(\tilde{\alpha}) \leq \beta < \tau^+\} = \{x_\beta^\theta : \varphi(\tilde{\alpha}) \leq \beta < \tau^+\}$ is a discrete τ-compact (by I.4) space of cardinality τ^+ - which is a contradiction. Hence $F_{\varphi(\tilde{\alpha})}^\theta \neq \emptyset$ for every $\theta < \theta^*$. Thus $\tilde{C}$ is a chain of non-empty closed sets in the τ-compact $F_{\varphi(\tilde{\alpha})}^0$. It follows that $\Phi_{\tilde{\alpha}} \neq \emptyset$. We choose $y_{\tilde{\alpha}}$ to be any point of $\Phi_{\tilde{\alpha}}$. Then $y_{\tilde{\alpha}} \in c\ell_\tau(\xi_0)$. Let $\eta = \{y_\alpha : \alpha < \tau^+\} \subset X$ be defined in accordance with the rule described. If $\alpha \leq \beta < \tau^+$, then $\varphi(\alpha) \leq \varphi(\beta)$ and $y_\beta \in F_{\varphi(\beta)}^0 \subset F_{\varphi(\alpha)}^0$. Hence $c\ell\{y_\beta : \alpha \leq \beta < \tau^+\} \subset c\ell\, F_{\varphi(\alpha)}^0$. On the other hand, $\{y_\beta : \beta < \alpha\} \subset cl\{x^o_\beta : \beta < \varphi(\alpha)\}$. As ξ_o is a free sequence in X, we have: $cl\{x^o_\beta : \beta < \varphi(\alpha)\} \cap$ $\cap\, cl\, F^o_{\varphi(\alpha)} = \emptyset$. Hence $cl\{y_\beta : \beta < \alpha\} \cap cl\{y_\beta :$ $: \alpha \leq \beta < \tau^+\} = \emptyset$ - i.e. η is a free sequence in X.

Let us show that $\eta < \xi_\theta$ for each $\theta < \theta^*$. If $\alpha \leq \beta < \tau^+$ then $y_\beta \in \Phi_\beta \subset F^\theta_{\varphi(\beta)}$ and $\alpha \leq \varphi(\alpha) \leq$ $\leq \varphi(\beta)$. Hence $F^\theta_{\varphi(\beta)} \subset F^\theta_{\varphi(\alpha)} \subset F^\theta_\alpha$ and $\{y_\beta : \alpha \leq \beta <$ $< \tau^+\} \subset F^\theta_\alpha = (cl_\tau\{x^\theta_\beta : \alpha \leq \beta < \tau^+\}) \setminus \{x^\theta_\beta : \alpha \leq \beta < \tau^+\}$, for each $\alpha < \tau^+$. By I.1, this means that $\eta < \xi_\theta$. Thus $\xi_{\theta*} = \eta$ is the required free sequence in X. Lemma I.5 is proved.

I.6. Definition. For a cardinal τ, $\mathcal{C}_\tau$ is the class of all spaces X satisfying the following condition: if Y is a discrete subspace of X and $|Y| \geq \tau^+$ then there exists $Z \subset Y$ such that $|Z| = \tau^+$ and Z is closed in X.

I.7. Proposition. If $\psi(F,X) \leq \tau$ for every closed set F in X then $X \in \mathcal{C}_\tau$.

Proof. Let Y be a discrete subspace of X such that $|Y| = \tau^+$. Then the set $F = cl(Y) \setminus Y$ is closed in X. Hence there exists a family γ of open sets in X such that $|\gamma| \leq \tau$ and $\bigcap \gamma = F$. From $Y \cap F = \emptyset$ it follows that $Y = \bigcup \{(X \setminus \mathcal{U}) \cap Y : \mathcal{U} \in \gamma\}$. As $|\gamma| \leq \tau < \tau^+ \leq |Y|$, there exists $\mathcal{U}^* \in \gamma$ such that $|Y \cap (X \setminus \mathcal{U}^*)| \geq \tau^+$. Clearly, every subset of the set $Y \cap (X \setminus \mathcal{U}^*)$ is closed in X. Thus any $Z \subset Y \cap (X \setminus \mathcal{U}^*)$ such that $|Z| = \tau^+$ is what we look for.

I.8. Corollary. If X is a space with countable development (i.e. a Moore space) then $X \in \mathcal{C}_\tau$ for every $\tau \geq \aleph_0$.

I.9. <u>Corollary</u>. If X is hereditarily Lindelöf then $X \in \mathcal{C}_\tau$ for each $\tau \geq \aleph_0$.

I.10. <u>Proposition</u>. Let X be a τ-compact and $X \subset Y$. Then either $t(y,M) \leq \tau$ (see [4]) for each $y \in Y$ and each $M \subset X$ such that $y \in cl\, M$, or there exists in X a free sequence of the length τ^+ .

Proof. Assume that $A \subset X$, $y^* \in Y$ and $y^* \in cl(A) \setminus cl_\tau(A)$ (here and in what follows the both closures are taken with respect to Y). Put $A^* = cl_\tau(A) \cap X$. Obviously, $y^* \notin cl_\tau(A^*)$. Let γ be a family of open sets in Y such that $|\gamma| \leq \tau$ and $\bigcap \gamma \ni y^*$. We shall prove that $(\bigcap \gamma) \cap A^* \neq \emptyset$. For each $\mathcal{U} \in \gamma$ we fix an open set $V_{\mathcal{U}}$ in Y such that $y^* \in V_{\mathcal{U}} \subset cl(V_{\mathcal{U}}) \subset \mathcal{U}$. Put $\gamma' = \{cl(V_{\mathcal{U}}) : \mathcal{U} \in \gamma\}$. As $|\gamma'| \leq |\gamma| \leq \tau$, we can write: $\gamma' = \{F_\alpha : \alpha < \tau\}$. Obviously $F_0 \cap A^* \neq \emptyset$ and $cl(F_0 \cap A^*) \ni y^*$. We are going to prove that $cl(\bigcap\{F_\alpha : \alpha \leq \beta\} \cap A^*) \ni y^*$ for each $\beta \leq \tau$. Suppose that this is true for every $\beta < \beta^*$, for some $\beta^* \leq \tau$. Then $C = \{\bigcap\{F_\alpha : \alpha \leq \beta\} \cap A^* : \beta < \beta^*\}$ is a chain of non-empty closed sets in the space A^* , each of which contains y^* in its closure. Let Oy^* be an arbitrary neighborhood of y^* in Y. Clearly every element of C intersects the set $F_{\beta^*} \cap cl(Oy^*)$. Thus $C' = \{\bigcap\{F_\alpha : \alpha \leq \beta\} \cap F_{\beta^*} \cap cl(Oy^*) \cap A^* : \beta < \beta^*\}$ is a chain of non-empty closed sets in A^* . By I.4, A^* is a τ-compact. Hence, in view of $|C'| \leq |\beta^*| \leq \tau$, $\bigcap C' \neq \emptyset$.
We have: $\bigcap C' = \bigcap\{F_\alpha : \alpha \leq \beta^*\} \cap A^* \cap cl(Oy^*)$.

As the space Y is regular and Oy^* is any neighborhood of y^*, it follows that $cl(\bigcap\{F_\alpha : \alpha \leq \beta^*\} \cap A^*) \ni y^*$. The transfinite induction is complete. Put $\beta = \tau$ into the last formula. We obtain: $\bigcap\{F_\alpha : \alpha \leq \tau\} \cap A^* \neq \emptyset$. Hence $\bigcap \gamma \cap A^* \neq \emptyset$. Now we can apply the fundamental lemma 4 from [3] to Y, y^* and A^*. It follows that there exists in Y a free sequence ξ of the length τ^+ such that $\xi \subset A^*$. But $A^* \subset X$. Hence $\xi \in \mathcal{F}_\tau(X)$.

I.11. Proposition. Let X be a τ-compact, $\tau \geq \aleph_0$ and $X = \bigcup\{X_\alpha : \alpha < \tau\}$, where $X_\alpha \in \mathcal{E}_\tau$ for every $\alpha < \tau$. Then there exists no free sequence in X of the length τ^+.

Proof. Let us assume that $\xi = \{x_\alpha : \alpha < \tau^+\} \in \mathcal{F}_\tau(X)$.

For each $\alpha \leq \tau$ we shall define $\eta_\alpha \in \mathcal{F}_\tau(X)$ under the following restrictions: 1) if $\alpha' < \alpha'' \leq \tau$ then $\eta_{\alpha''} < \eta_{\alpha'}$, and 2) if $\alpha < \tau$, $\eta \in \mathcal{F}_\tau(X)$ and $\eta \leq \eta_\alpha$ then $|\eta \cap X_{\alpha'}| \leq \tau$ for each $\alpha' < \alpha$.

We put $\eta_0 = \xi$. Let $\beta^* \leq \tau$ and assume that $\eta_\alpha \in \mathcal{F}_\tau(X)$ is defined for every $\alpha < \beta^*$ in such a way that the conditions 1) and 2) are satisfied for all these α. If β^* is a limit ordinal we choose η_{β^*} to be any $\eta' \in \mathcal{F}_\tau(X)$ such that $\eta' < \eta_\alpha$ for all $\alpha < \beta^*$ (see I.5). Suppose now that β^* has an immediate predecessor α^*. If there exists no $\eta \in \mathcal{F}_\tau(X)$ such that $\eta < \eta_{\alpha^*}$ and $|\eta \cap X_{\alpha^*}| = \tau^+$, then we choose η_{β^*} to be any $\eta' \in \mathcal{F}_\tau(X)$ such that $\eta' < \eta_{\alpha^*}$ (see I.5). Let us assume now that there exists $\eta \in \mathcal{F}_\tau(X)$ such that $\eta < \eta_{\alpha^*}$ and $|\eta \cap X_{\alpha^*}| = \tau^+$. We fix such η.

The set $\eta \cap X_{\alpha *}$ is discrete, $|\eta \cap X_{\alpha *}| = \tau^+$ and $X_{\alpha *} \in \mathcal{E}_\tau$. Hence there exists a set $Z \subset \eta \cap X_{\alpha *}$ closed in $X_{\alpha *}$ such that $|Z| = \tau^+$. Obviously there exists $\eta' \in \mathcal{F}_\tau(X)$ such that $\eta' = Z$ and elements in η' are ordered in the same way as they are ordered in η. By I.5, there exists $\eta'' \in \mathcal{F}_\tau(X)$ such that $\eta'' < \eta'$. Then $\eta'' \cap X_{\alpha *} = \emptyset$ and if $\eta''' \leq \eta''$, then $\eta''' \cap \cap X_{\alpha *} = \emptyset$. Indeed $\eta''' \leq \eta''$ implies that $\eta''' < \eta'$ from which it follows that $\eta''' \subset c\ell(\eta') \setminus \eta' \subset X \setminus X_{\alpha *}$. We put $\eta_{\beta *} = \eta''$. Clearly the conditions 1) and 2) are satisfied for all $\alpha \leq \beta^*$. Thus a transfinite sequence $\{\eta_\alpha : \alpha \leq \tau\} \subset \mathcal{F}_\tau(X)$ satisfying the conditions 1) and 2) exists. Let us fix it. Consider η_τ. From 2) it follows that $|\eta_\tau \cap X_\alpha| \leq \tau$ for every $\alpha < \tau$. Hence $|\eta_\tau \cap X| \leq \tau$. But $\eta_\tau = \eta_\tau \cap X$ implies that $|\eta_\tau \cap X| = |\eta_\tau| = \tau^+$. The contradiction we arrived at means that $\mathcal{F}_\tau(X) = \emptyset$.

Now we are ready to formulate and prove one of our main results.

I.12. Theorem. If X is a τ-compact, $\tau \geq \aleph_0$ and $X = \bigcup\{X_\alpha : \alpha < \tau\}$, where $X_\alpha \in \mathcal{E}_\tau$ for every $\alpha < < \tau$, then $t(X) \leq \tau$. In particular, if X is a compact and each X_α is developable then $t(X) \leq \tau$.

Proof. We just apply I.11 and I.10 where Y = X.

I.13. Definition. A space X is called τ-bounded if for every $A \subset X$, such that $|A| \leq \tau$, $c\ell(A)$ is compact.

I.14. Theorem. Let X be a τ-bounded completely regular space, $\tau \geq \aleph_0$ and $X = \bigcup\{X_\alpha : \alpha < \tau\}$ where

$X_\alpha \in \mathcal{C}_\tau$ for every $\alpha < \tau$. Then X is compact and $t(X) \le \tau$.

Proof. In view of I.12 we only have to prove that X is compact. Let us fix a compact extension Y of the space X. Let $y \in Y$. From I.10 it follows that $t(y,X) \le \tau$. Hence $y \in c\ell(A)$ for some $A \subset X$ such that $|A| \le \tau$. As X is τ-bounded, $c\ell(A) \cap X$ is compact. Hence $c\ell(A) \cap X$ is closed in Y. This implies that $c\ell(A) = c\ell(A) \cap X$. Thus $y \in c\ell(A) \subset X$, i.e. X = Y. We conclude that X is compact.

I.15. <u>Notations</u>. S_τ is the class of all spaces X such that $X = \cup\{X_\alpha : \alpha < \tau\}$ where $X_\alpha \in \mathcal{C}_\tau$ for every $\alpha < \tau$. By $\mathcal{M}_\tau$ we denote the class of all X such that $X = \cup\{X_\alpha : \alpha < \tau\}$ where X_α is metrizable for every $\alpha < \tau$. We put $\mathcal{M}^* = \cup\{\mathcal{M}_n : n \in N^+\}$.

Straight from I.12 we get

I.16. <u>Theorem</u>. If X is a k-space, $\tau \ge \aleph_0$ and $X \in S_\tau$ then $t(X) \le \tau$.

I.17. <u>Observation</u>. If X is a space of point countable type and $X \in \mathcal{M}_{\aleph_0}$, then X is first countable at a dense set of points.

This follows trivially from the fact that every compact is of second category.

The following assertion provides us with additional strong information on the structure of compacts belonging to $\mathcal{M}_{\aleph_0}$.

I.18. <u>Theorem</u>. If X is a τ-compact, $\tau \ge \aleph_0$ and $X \in \mathcal{M}_{\aleph_0}$, then the following conditions are pairwise equivalent: a) $c(X) \le \tau$; b) $d(X) \le \tau$; c) for every

$Y \subset X$ such that $c\ell(Y) = X$, $d(Y) \le \tau$; d) $\pi w(X) \le \tau$; e) there exists $Y \subset X$ such that $c\ell(Y) = X$ and $w(Y) \le \tau$.

Proof. It is well known that e) $\Longrightarrow$ d) $\Longrightarrow$ c) $\Longrightarrow$ b) $\Longrightarrow$ a). It remains to show that a) $\Longrightarrow$ e). Let $X = \bigcup \{ Y_i : i \in N^+ \}$ where each Y_i is metrizable. Assume that $c(X) \le \tau$. Put $\gamma = \{ \mathcal{U} \subset X : \mathcal{U}$ is open in X, $\mathcal{U} \ne \emptyset$ and $\mathcal{U} \cap Y_i$ is dense in $\mathcal{U}$ for some $i \in N^+ \}$.
As X is of second category, γ is a π-base of X. There exists a maximal disjoint subfamily γ^* of the family γ . Then $c\ell(\bigcup \gamma^*) = X$. We have: $|\gamma^*| \le c(X) \le \tau$. For each $\mathcal{U} \in \gamma^*$ we choose $i \in N^+$ such that $\mathcal{U} \cap Y_i$ is dense in $\mathcal{U}$ and put $Z_{\mathcal{U}} = \mathcal{U} \cap Y_i$. As $\mathcal{U}$ is open in X, $c(\mathcal{U}) \le c(X) \le \tau$. As $Z_{\mathcal{U}}$ is dense in $\mathcal{U}$, $c(Z_{\mathcal{U}}) \le c(\mathcal{U}) \le \tau$. As $Z_{\mathcal{U}}$ is metrizable, it follows that $w(Z_{\mathcal{U}}) = c(Z_{\mathcal{U}}) \le \tau$. We put $Z = \bigcup \{ Z_{\mathcal{U}} : \mathcal{U} \in \gamma^* \}$. As γ^* is disjoint, $\mathcal{U} \cap Z = Z_{\mathcal{U}}$. Hence $Z_{\mathcal{U}}$ is open in Z for every $\mathcal{U} \in \gamma^*$. Let $\mathcal{B}_{\mathcal{U}}$ be a base of $Z_{\mathcal{U}}$ such that $|\mathcal{B}_{\mathcal{U}}| = w(Z_{\mathcal{U}}) \le \tau$. Then $\mathcal{B} = \bigcup \{ \mathcal{B}_{\mathcal{U}} : \mathcal{U} \in \gamma^* \}$ is a base of the space Z and $|\mathcal{B}| \le \tau \cdot \tau = \tau$. Hence $w(Z) \le \tau$. Obviously, $c\ell(Z) = c\ell(\bigcup \gamma^*) = X$.

§ 2. The case of two summands

II.1. Example. Let $\tau \ge \aleph_0$ We fix a discrete space A_τ such that $|A_\tau| = \tau$. Denote by A^*_τ the compact extension of A_τ by one point: $A^*_\tau = A_\tau \cup \{ \xi_\tau \}$. Then A^*_τ is the union of two discrete subspaces. Hence $A^*_\tau \in \mathcal{M}_\tau$. On the other hand, A^*_τ is not first countable at the point ξ_τ if $\tau > \aleph_0$. Thus not every compact belonging to

$\mathcal{M}_2$ is metrizable.

In this paragraph we study the compact elements of $\mathcal{M}_2$ in greater detail.

II.2. Lemma. Let $\tau \geq \aleph_0$, X is τ-compact, $X = X_1 \cup \cup X_2$ and assume that the following conditions are satisfied for $i = 1,2$: a) if $A \subset X_i$ and $|A| \leq \tau$ then the closure of A in X_i has a network γ_i such that $|\gamma_i| \leq \tau$; b) $\psi(X_i) \leq \tau$. Then for every $A \subset X_1$ such that $|A| \leq \tau$ and for each $z \in c\ell(A) \cap X_2$, $\chi(z, c\ell(A)) \leq \tau$.

Proof. We put $A_i = c\ell(A) \cap X_i$ and fix $z \in A_2$. It follows from a) that there exists a family γ of sets in A_1 such that $|\gamma| \leq \tau$, $\bigcup\gamma \supset A_1 \setminus \{x\}$ and $c\ell(P) \not\ni z$ for every $P \in \gamma$. We put $\widetilde{\gamma}_1 = \{c\ell(A) \setminus c\ell(P) : P \in \gamma\}$. Then $|\widetilde{\gamma}_1| \leq |\gamma| \leq \tau$ and $(\bigcap \widetilde{\gamma}_1) \cap X_1 \subset \{x\}$. As $\psi(z, A_2) \leq \psi(z, X_2) \leq \tau$, there exists a family $\widetilde{\gamma}_2$ of open sets in $c\ell(A)$ such that $|\widetilde{\gamma}_2| \leq \tau$ and $(\bigcap \widetilde{\gamma}_2) \cap \cap X_2 = \{z\}$. Let $\widetilde{\gamma} = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$. Then obviously $|\widetilde{\gamma}| \leq \tau$ and $\bigcap \widetilde{\gamma} = \{x\}$. As all elements of $\widetilde{\gamma}$ are open sets in $c\ell(A)$, it follows that $\psi(x, c\ell(A)) \leq |\widetilde{\gamma}| \leq \tau$. But $c\ell(A)$ is τ-compact. Hence $\chi(x, c\ell(A)) \leq \psi(x, c\ell(A)) \leq \tau$.

II.3. Proposition. Let X be an $\aleph_0$-compact and $X = X_1 \cup X_2$ where X_i, for each $i = 1,2$, satisfy the following conditions: 1) $X_i \in \mathcal{C}_{\aleph_0}$; 2) if $A \subset X_i$ and $|A| \leq \aleph_0$ then the closure of A in X_i has a countable network; 3) $\psi(X_i) \leq \aleph_0$. Then, for every $A \subset X$, $c\ell(A) = \operatorname{seq} c\ell(\operatorname{seq} c\ell(A))$ (and hence X is sequential).

Proof. By Theorem I.12, $t(X) \leq \aleph_0$. We fix $A \subset X$ and $x \in c\ell(A)$. There exists $B \subset A$ such that $|B| \leq \aleph_0$ and

$x \in c\ell(B)$. We put $B_i = B \cap X_i$, $i = 1,2$. Then either $x \in c\ell(B_1)$ or $x \in c\ell(B_2)$. It is sufficient to consider the case when $x \in c\ell(B_1)$. There are two possibilities: I) $x \in X_2$, and II) $x \notin X_2$. If I) holds, we apply the lemma II.2 with $A = B_1$ and conclude that $\chi(x, c\ell(B_1)) \leq \aleph_0$. Hence $x \in \mathrm{seqc}\ell(B_1) \subset \mathrm{seqc}\ell(A)$. It remains to consider the case II): $x \notin X_2$. Then $x \in X_1$. We put $C = c\ell(B_1)$ and $C_1 = C \cap X_2$. It is necessary to distinguish the two following subcases: II_1) $c\ell(C_1) \not\ni x$ and II_2) $c\ell(C_1) \ni x$. Let $c\ell(C_1) \not\ni x$. Then there exists a neighborhood Ox of x in X such that $c\ell(C_1) \cap c\ell(Ox) = \emptyset$. Then $F = c\ell(Ox) \cap C$ is an $\aleph_0$-compact and $F \cap X_2 = \emptyset$. Hence $F \subset X_1$. From $x \in F$ and $\psi(F) \leq \psi(X_1) \leq \aleph_0$ it follows then that $\chi(x,F) \leq \aleph_0$. We have: $B_1' = Ox \cap B_1 \subset F$ and $x \in c\ell(B_1')$. Hence there exists a sequence in B_1' converging to x. As $B_1' \subset B \subset A$, we conclude: $x \in \mathrm{seqc}\ell(A)$ - and the proof in the case II_1) is complete. Suppose now that II_2) holds: $c\ell(C_1) \ni x$. As $t(X) \leq \aleph_0$, there exists a countable set $C_1^* \subset C_1$ such that $x \in c\ell(C_1^*)$. Then we have: $C_1^* \subset X_2$, $|C_1^*| \leq \aleph_0$ and $x \in X_1$, $x \in c\ell(C_1^*)$. Hence we can apply the lemma II.2 (where X_1 plays the role of X_2 and X_2 plays the role of X_1). It follows that $\chi(x, c\ell(C_1^*)) \leq \aleph_0$. Thus $x \in \mathrm{seqc}\ell(C_1^*)$. From $B_1 \subset X_1$ and $|B_1| \leq \aleph_0$ it follows by lemma II.2 that $c\ell(B_1)$ is first countable at all the points of the set $c\ell(B_1) \cap X_2 = C_1$. Hence $C_1^* \subset C_1 \subset \mathrm{seqc}\ell(B_1)$, so that $x \in \mathrm{seqc}\ell(C_1^*) \subset \mathrm{seqc}\ell(\mathrm{seqc}\ell(B_1)) \subset \mathrm{seqc}\ell(\mathrm{seqc}\ell(A))$. Proposition II.3 is proved.

Remark. The spaces X_1 and X_2 above need not be sequential.

II.4. Proposition. Let $X = X_1 \cup X_2$, where all the conditions from II.3 are satisfied. In addition, let us assume that for each $A \subset X_i$ such that $|A| \le \aleph_0$ the closure of A in X_i is a Fréchet-Urysohn space, $i = 1,2$. Then X is also a Fréchet-Urysohn space.

Proof. By Theorem I.12, $t(X) \le \aleph_0$. Hence it suffices to show that $c\ell(A) = \operatorname{seqc}\ell(A)$ for every countable $A \subset X$. Assume that $x \in c\ell(A) \setminus \operatorname{seqc}\ell(A)$. Let $x \in X_1$. As X_1 is Fréchet-Urysohn, $\operatorname{seqc}\ell(A \cap X_1) \supset c\ell(A \cap X_1) \cap X_1$. From $x \notin \operatorname{seqc}\ell(A) \supset \operatorname{seqc}\ell(A \cap X_1)$ it follows now that $x \notin c\ell(A \cap X_1)$. Thus $x \in c\ell(A \cap X_2)$. We have: $A \cap X_2 \subset X_2$ and $|A_m \cap X_2| \le \aleph_0$. From Lemma II.2 we conclude now that $\chi(x, c\ell(A \cap X_2)) \le \aleph_0$. Hence $x \in \operatorname{seqc}\ell(A \cap X_2) \subset \operatorname{seqc}\ell(A)$ - in contradiction with $x \in c\ell(A) \setminus \operatorname{seqc}\ell(A)$. Proposition II.4 is proved.

II.5. Proposition. Let $\tau \ge \aleph_0$. Assume that X is a τ-compact and $X = X_1 \cup X_2$, where, for each $i = 1,2$, the following conditions are satisfied: 1) $X_i \in \mathcal{E}_\tau$; 2) if $A_i \subset X_i$ and $|A_i| \le \tau$ then the weight of the closure of A_i in X_i does not exceed τ; 3) if $Y_i \subset X_i$ then either there exists a discrete subspace $Z_i \subset Y_i$ such that $|Z_i| = \tau^+$ (i.e. $s(Y_i) > \tau$) or the density $d(Y_i)$ of Y_i is not greater than τ.

Then for any $A \subset X$ such that $|A| \le \tau$, $w(c\ell(A)) \le \tau$.

Proof. Put $A_i = A \cap X_i$, $X_i^+ = c\ell(A_i)$ and $\widetilde{X}_i = X_i \cap X_i^+$, $i = 1,2$. By 2), $w(\widetilde{X}_i) \le \tau$. We have: $c\ell(A_i) =$

$= c\ell(\widetilde{X}_i) = X_i^+$ - i.e. $\widetilde{X}_i$ is dense in X_i^+. As X_i^+ is regular it follows that, for every compact $\Phi \subset \widetilde{X}_i$, $\chi(\Phi, X_i^+) = \chi(\Phi, \widetilde{X}_i) \le w(\widetilde{X}_i) \le \tau$. We put $X_i^- = X_i^+ \setminus \widetilde{X}_i$. Let us show that $d(X_i^-) \le \tau$ (density of X_i^- does not exceed τ).

Assume the contrary. Clearly, $X_i^- \subset X \setminus X_i$. Then, by 3) and 1), there exists a closed in X_i^- discrete set $D_i \subset X_i^-$ such that $|D_i| = \tau^+$. Put $\Phi_i = c\ell(D_i) \setminus D_i$. Then Φ_i is closed in X and hence Φ_i is a τ-compact. We observe that $D_i \subset X_i^+$ and X_i^+ is closed in X. Hence $\Phi_i \subset X_i^+$. As D_i is closed in X_i^-, it follows that $\Phi_i \cap X_i^- = \emptyset$. Thus $\Phi_i \subset \widetilde{X}_i$. Hence $w(\Phi_i) \le w(\widetilde{X}_i) \le \tau$. We can conclude now that Φ_i is a compact. It follows that $\chi(\Phi_i, X_i^+) \le \tau$. There exists a family γ_i of open sets in X_i^+ such that $\bigcap \gamma_i = \Phi_i$ and $|\gamma_i| \le \tau$. From $\Phi_i \cap D_i = \emptyset$ we obtain now that $D_i = \bigcup \{D_i \setminus \mathcal{U} : \mathcal{U} \in \gamma_i\}$. We have: $|\gamma_i| \le \tau$ and $|D_i| = \tau^+ > \tau$. Thus $|D_i \setminus \mathcal{U}_i| = \tau^+$ for some $\mathcal{U}_i \in \gamma_i$. As $D_i \setminus \mathcal{U}_i = c\ell D_i \setminus \mathcal{U}_i$, the set $D_i \setminus \mathcal{U}_i$ is closed in $c\ell D_i$ and hence $D_i \setminus \mathcal{U}_i$ is closed in X. But, as X is τ-compact, there exists no closed discrete set in X of cardinality τ^+. The contradiction we arrived at implies that $d(X_i^-) \le \tau$. Clearly, if $i = 1$ then $X_i^- \subset X_2$ and if $i = 2$ then $X_i^- \subset X_1$. In any case it follows from 2) and $d(X_i^-) \le \tau$ that $w(X_i^-) \le \tau$. Thus $c\ell(A) = \widetilde{X}_1 \cup \widetilde{X}_2 \cup X_1^- \cup X_2^-$, where $w(\widetilde{X}_i) \le \tau$, $w(X_i^-) \le \tau$, $i = 1,2$. The space $c\ell(A)$ is τ-compact, as X is τ-compact. From the theorem on the addition of weights proved in [7] (see also [9]) it follows now that $w(c\ell(A)) \le \tau$.

We can now formulate and prove the main results of this paragraph.

II.6. <u>Theorem</u>. If X is an $\aleph_0$-compact and $X = X_1 \cup X_2$ where X_1 and X_2 are spaces with uniform base then a) X is Fréchet-Urysohn, and b) for each countable $A \subset X$, $w(c\ell(A)) \leq \aleph_0$.

Proof. The assertion follows immediately from II.5.

II.7. <u>Corollary</u>. If X is a separable $\aleph_0$-compact and $X = X_1 \cup X_2$ where X_1 and X_2 are spaces with uniform base then $w(X) \leq \aleph_0$ (and hence X is a compact).

II.8. <u>Theorem</u>. Let X be a compact and $X = X_1 \cup X_2$ where X_1 and X_2 are spaces with uniform base. Then a) X is Fréchet-Urysohn, and b) for every $A \subset X$, $w(c\ell(A)) \leq |A|$. In other words, X is an exact compact in the sense of [6].

Proof. One should only observe that the conditions of II.5 are satisfied by X, X_1 and X_2 for all $\tau \geq \aleph_0$.

II.9. <u>Example.</u> Franklin's compact (see [8], or [1]) is the union of two separable locally metrizable developable spaces while it is not Fréchet-Urysohn. It does not satisfy b) in II.8 as well. Hence II.6, II.7 and II.8 are not extendable to the class of all developable spaces.

Observe that the case when X_1 and X_2 are metrizable is covered by II.6, II.7 and II.8. But in this case the assertion II.7 can be considerably strengthened.

II.10. <u>Theorem</u>. If X is a τ-compact and $X = X_1 \cup X_2$ where X_1 and X_2 are metrizable then the following conditions are equivalent: 1) $c(X) \leq \tau$ and 2) $w(X) \leq \tau$.

Proof. This follows from I.18 and II.5.

II.11. Corollary. If X is a compact and $X = X_1 \cup \cup X_2$ where X_1 and X_2 are metrizable then $c(A) = w(A)$ for every $A \subset X$ (and X is Fréchet-Urysohn by II.8).

II.12. Example. The same Franklin's compact (see II.9) is the union of three discrete spaces while it is separable and not metrizable and not Fréchet-Urysohn. Hence none of the results II.6, II.7, II.8 and II.10 can be generalized to the case of the union of three metrizable spaces. That is the real reason why we had to consider the case of two summands separately. We shall treat the peculiarities of the case when a compact is the union of finitely many metrizable spaces in our next paper. We would like to conclude with the following problem, motivated by II.8 and II.10.

II.13. Problem. Is it true that every compact which can be represented as the union of two metrizable subspaces is an Eberlein compact?

I also want to formulate here the following problem which was posed in my talk at the Prague Symposium, 1976 and was recently solved by A. Ostaszewski (in the affirmative).

II.14. Problem. Let X be a compact such that $X \in \in \mathcal{M}_{\aleph_0}$. Is it true then that X is sequential?

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