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A NONLINEAR OPERATOR IN POTENTIAL THEORY

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Abstract: A property of the first eigenvalue of the operator Δ leads to the solvability of a nonlinear equation whose main part is a singular linear equation.

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1. Let T be the nonlinear operator defined by $T(u) = (\Delta + c^2)u + p(u)$, where Δ is the Laplacean in the unit disk D: $x^2 + y^2 < 1$, p is continuous on $(-\infty, \infty)$, and $p(u) = o(|u|)$, $\lim |p(u)| = +\infty$, as $|u| \longrightarrow \infty$. The domain of the operator Δ is the space of all u, continuous on D, vanishing on ∂D, whose Laplacean (in the sense of distributions) belongs to $L^2(D)$; Green's formula confirms that these functions u are Hölder-continuous. Moreover $-c^2$ is the smallest eigenvalue of this operator, and Δ is a closed, negative-definite operator.

Theorem. For each $r > 2$ and each M, the set defined by the inequality $\| T(u) \|_r \leq M$ is compact in the Banach space $C^1(D^-)$, and the range of T meets $L^r(D)$ in a closed subset of $L^r(D)$.

The range contains $L^r(D)$ if and only if

$p(+\infty)\cdot p(-\infty)<0$.

This theorem was suggested by a remarkable paper of Ambrosetti and Prodi [1] in which a similar use is made of the first eigenvalue of the operator Δ .

2. The operator $\Delta + a^2$ is singular precisely when $a>0$ is a zero of some Bessel function J_k, and the eigenfunction for c^2 is $f_0 = J_0(cr)$; $f_0>0$ within D, and the normal derivative of f_0 is negative on ∂D. (See [2, p. 373].) (Tables show that $c \cong 2.40$ and the next zero is $\cong 3.83$.)

Green's formula, with zero boundary data,

$$f(z) = (2\pi)^{-1}\iint (\Delta f)(z')\, G(z,z')dx'dy'$$

shows that if $\Delta f \in L^r$, $2<r<\infty$, then $f \in C^1(D^-)$, and the first partial derivatives of f are Hölder-continuous in exponent $1 - 2/r$;this is proved by means of Hölder's inequality and the potential-theoretic lemmas presented in [3, p. 198]. When $1<r<2$ similar consideration yields Hölder-continuity of f.

To prove the first statement in the theorem we take a sequence u_n, in the domain of Δ , such that $\| T(u_n)\|_r \leq M$. Supposing that $m_n = \|u_n\|_\infty$ tends to infinity, we proceed to obtain a contradiction. We write $u_n = a_n f_0 + v_n$, where v_n is orthogonal to f_0 in $L^2(D)$, and a_n is a real number. Since $p(u) = o(|u|)$ as $|u|\to +\infty$, we see that $(\Delta + c^2)v_n = o(m_n)$ uniformly, and therefore in L^2. By the discreteness of the spectrum of Δ , we see that $(\Delta + c^2)v_n$ and Δv_n are of the same magnitude in L^2, whence $v_n = o(m_n)$ <u>uniformly</u>. We now observe the identities

$$\Delta v_n = (\Delta + c^2)v_n - c^2 v_n = T(u_n) - p(u_n) + c^2 v_n,$$

and deduce that $\Delta v_n = o(m_n)$ in $L^r(D^2)$. Therefore $v_n = o(m_n)$ in the Banach space $C^1(D^-)$, whence $v_n(z) = o(m_n)(1 - |z|)$. We have also $a_n \simeq \pm m_n$, so that $u_n = a_n f_0 + v_n$ has <u>no zeroes</u> for large n, in view of the inequality $f_0(z) \geq a(1 - |z|)$ valid for some $a > 0$.

If, for example, $a_n > 0$ and $p(+\infty) = +\infty$, then $p(u_n)$ tends everywhere to $+\infty$, while $p(u_n) \geq -C$. But $(\Delta + c^2)u_n$ is orthogonal to f_0, so $\iint p(u_n)f_0(z)dxdy = O(1)$, while $f_0 > 0$. This contradiction shows that m_n must remain bounded.

Now, by steps similar to the above, we find that $a_n = O(1)$, so $\| \Delta v_n \|_r = O(1)$, and then the functions u_n are bounded, with uniformly Hölder-continuous partial derivatives, in exponent $1 - 2/r$.

To prove the closure of the range of T in L^r, suppose $\lim T(u_n) = g$ in L^r; we can then select a subsequence u_j, converging to u_0 in $C^1(D^-)$. Now $\Delta u_j = T(u_j) - c^2 u_j - p(u_j)$ and Green's formula shows that $\Delta u_0 = g - c^2 u_0 - p(u_0)$, or $T(u_0) = g$.

3. Suppose now that $p(u) \geq -C$ for all u; then $(T(u), f_0) \geq -C'$, so that the range of T contains λf_0 only

when $\lambda \leq \lambda_0$.

To complete the proof, we suppose that $p(+\infty) = +\infty$ and $p(-\infty) = -\infty$ and prove that $T(u) = g$ is solvable for every g in L^r, $r > 2$. First we solve a perturbed equation $T(u) + \varepsilon u = g$, for small $\varepsilon > 0$. We write this in the form

$$(\Delta + c^2 + \varepsilon)u = g - p(u)$$

and observe that $\Delta + c^2 + \varepsilon$ admits a bounded completely continuous inverse in L^2, for small ε. Let us define

$$A_\varepsilon(u) = (\Delta + c^2 + \varepsilon)^{-1}(g - p(u)).$$

A_ε is continuous because $g \in L^2$ and $p(u) = o(|u|)$, and compact, because $(\Delta + c^2 + \varepsilon)^{-1}$ is compact. On the ball $\|u\|_2 \leq N$, we have $\|A_\varepsilon(u)\|_2 = o(N)$ so that A_ε is a compact mapping of some ball into itself and admits a fixpoint by Schauder's theorem, i.e. a solution of the perturbed equation. To obtain a solution to the original equation, we prove that the solutions of the equations $(\Delta + c^2 + \varepsilon)u + p(u) = g$ remain bounded as $\varepsilon \longrightarrow 0+$. We write $u = a_\varepsilon f_0 + v_\varepsilon$, and suppose that $\|u\|_\infty$ becomes unbounded. Then $\|v_\varepsilon\|_2 = o(1)\|u\|_\infty$, and we observe that

$$\Delta v_\varepsilon = g - p(u) - c^2 v_\varepsilon - \varepsilon a_\varepsilon f_0.$$

Thus $\|v_\varepsilon\|_\infty = o(1)\|u\|_\infty$, and finally $v_\varepsilon = o(1)\|u\|_\infty$ in $C^1(D^-)$. Hence u maintains the same sign, and $\varepsilon u + p(u)$ tends to $+\infty$ (or $-\infty$) remaining bounded below (above), so that the inner produce $(T_\varepsilon u, f_0)$ becomes infinite. This completes the proof in the case $p(+\infty) > 0$, $p(-\infty) < 0$. In the event that $p(+\infty) < 0$, $p(-\infty) > 0$ we employ the perturbed operator $T(u) - \varepsilon u$.

4. **An extension.** The main theorem remains true in part for each $r > 1$, but to verify this we must consider the inverse of the operator $\Delta + c^2$ on the appropriate subspace of L^r. It seems likely that an existence theorem remains true when $r = 1$, provided p' is bounded; the analysis would be difficult since the solutions u are unbounded.

R e f e r e n c e s

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