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K-ESSENTIAL SUBGROUPS OF ABELIAN GROUPS II

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Abstract: The purpose of this paper is to continue the investigation of K-essential subgroups of abelian groups begun in [1]. There is given a generalization of the group-socle and the intersections of K-essential subgroups of a group G are investigated with respect to the existence of the smallest K-essential subgroup of G. The theorem 3.3 gives a description of the intersection of all the maximal K-essential subgroups (a generalization of the Frattini-subgroup). Finally, there is investigated the Galois-correspondence on the power-set of all subgroups of G defined by the relation "A is B-essential in G". Further, the notion of the pure-closure is generalized and the topologies of G defined by the filters of K-essential subgroups for various subgroups K of G are studied.

Key words: K-essential, maximal K-essential, essential subgroups; K-socles, socles, elementary groups; K-nongenerators, Frattini subgroups; $\mathfrak{N}$-closure and pure closure operators; essential topologies.

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0. Introduction. This paper develops the theory of K-essential subgroups as it was introduced in [1]. All groups considered here are abelian. Concerning the terminology and notation we refer to [3],[4] and [1]. For convenience, we are going to introduce the following definition from [1].

Definition: Let G be a group and K a subgroup of G. A subgroup N of G is said to be K-essential in G if for every $g \in G \setminus K$ there is an integer $n > 0$ with $ng \in N \setminus K$.

Notice that the set of all K-essential subgroups of G is a filter (see 1.4 [1]).

Let $K \subset N$ be subgroups of a group G. Following Krivonos [5], a subgroup A of G is said to be N-K-high in G if A is maximal with respect to the property $A \cap N = K$.

Denote by $\overline{\mathbb{N}}$ the set of all square-free integers.

1. The K-socle and K-essential subgroups.

Definition 1.1. Let K be a subgroup of a group G. The set of all $g \in G$ such that there is $n \in \overline{\mathbb{N}}$ with $ng \in K$ we call K-socle of G and denote by G^K.

Obviously, G^K is a subgroup of G containing K. Further, G^0 is the socle of G. The group G^K/K is the socle of G/K, i.e. $(G/K)^0 = G^K/K$. The subgroup G^K is generated by the family of all elements $g \in G$ that there is $p \in \mathbb{P}$ with $pg \in K$.

Lemma 1.2. Let K be a subgroup of a group G. Then for each element $g \in G \setminus G^K$ there exists a K-essential subgroup N of G with $G^K \subset N$ and $g \notin N$.

Proof. Let $g \in G \setminus G^K$ and p be a prime such that $\sigma(g + K) < \infty$ implies $p^2 \mid \sigma(g + K)$. Now, $g \notin \langle G^K, pg \rangle$. For, if $g = s + kpg$, where $s \in G^K$ and k is an integer, then $(kp - 1)g \in G^K$. Consequently, there is $n \in \overline{\mathbb{N}}$ such that $n(kp - 1)g \in K$. Hence $p^2 \mid n(kp - 1)$, a contradiction.

Let N be a subgroup of G maximal with respect to the properties: $\langle G^K, pg \rangle \subset N$, $g \notin N$. Then N is K-essential in G. For, if $x \in G \setminus K \cup N$ then $g \in \langle x, N \rangle$, i.e. $g = rx + n$, where $n \in N$ and r is an integer. Now, $prx = pg - pn \in N$. If $prx \in K$ then $rx \in G^K$ and $g \in N$, a contradiction. Hence $prx \in N \setminus K$.

Lemma 1.3. Let K and N be subgroups of a group G. Then

(i) N is K-essential in G containing K iff N is essential in G containing G^K;

(ii) If N is K-essential in G then N + K is an essential subgroup of G containing G^K.

Proof. (i) Let N be a K-essential subgroup of G containing K. If $g \in G$ then either $g \in K \subset N$ or there is $n \in \mathbb{N}$ such that $ng \in N \setminus K$. Hence N is essential in G. Let $g \in G \setminus N$ and $pg \in K$ for a prime p. Now, there is $k \in \mathbb{N}$ with $kg \in N \setminus K$; consequently $(p,k) = 1$. There are integers u,v such that $up + vk = 1$ and $g = upg + vkg \in N$, a contradiction. Hence $G^K \subset N$.

Let N be an essential subgroup of G containing G^K. Let $g \in G \setminus K$ and n be the least nonzero natural number with $ng \in N$. If $ng \in K$ then $n = pr$ for a prime p and a natural number r. Now, $rg \in G^K \subset N$ and $r < n$, a contradiction. Hence $ng \in N \setminus K$.

(ii) It follows from (i).

Proposition 1.4. Let K be a subgroup of a group G. The following are equivalent:

(i) $G^K = G$;

(ii) G/K is an elementary group;

(iii) If N is K-essential in G then $N + K = G$.

Proof. (i) $\Longrightarrow$ (iii) If N is K-essential in G then $G^K \subset N + K$ by 1.3. Hence $N + K = G$ by (i).

(iii) $\Longrightarrow$ (i) If $g \in G \setminus G^K$ then there is a K-essential subgroup N of G such that $G^K \subset N$ and $g \notin N$ by 1.2. Hence

$N + K = N \neq G$, a contradiction.

(i)$\Longleftrightarrow$(ii) It is trivial.

Corollary 1.5. A group G has no proper essential subgroups iff G is elementary.

Proposition 1.6. Let K and N be subgroups of a group G. Then the following are equivalent:

(i) K is $N - N\cap K$-high in G;

(ii) $N + K$ is K-essential in G;

(iii) $N + K$ is essential in G and $G^K \subset N + K$.

Proof. (i)$\Longrightarrow$(ii) If $g\in G\setminus K$ then $\langle g,K\rangle \cap N \supsetneqq N\cap K$, i.e. there are $n \in \mathbb{N}$, $k\in K$ and $m\in N\setminus K$ such that $ng + k = m$. Hence $ng \in (N + K)\setminus K$.

(ii)$\Longrightarrow$(i) If $g\in G\setminus K$ then there is $n \in \mathbb{N}$ such that $ng \in (N + K)\setminus K$. Hence $ng = m + k$, where $m\in N\setminus K$ and $k\in K$; consequently $\langle g,K\rangle \cap N \supsetneqq N\cap K$.

(ii)$\Longleftrightarrow$(iii) By 1.3.

Corollary 1.7. Let K and N be subgroups of a group G. Then K is N-high in G iff $K \oplus N$ is an essential subgroup of G containing G^K.

2. Intersections of K-essential subgroups.

Proposition 2.1. Let K be a subgroup of a group G. Then the K-socle of G is the intersection of all K-essential subgroups of G containing K.

Proof. It follows immediately from 1.2 and 1.3.

Definition 2.2. Let K be a subgroup of a group G. Write $G_K = \bigoplus_{p\in \mathbb{P}_K} (G_p)^{K_p}$, where $\mathbb{P}_K$ is the set of all primes p with $K_p \neq G_p$.

Theorem 2.3. Let K be a subgroup of a group G. Then the intersection of all K-essential subgroups of G is contained in the K-socle G^K of G and contains the group G_K.

Proof. The intersection of all K-essential subgroups of G is contained in G^K by 2.1.

Let N be a K-essential subgroup of G. If $p \in \mathbb{P}_K$ then there is $g \in G_p \setminus K$ and there exists $n \in \mathbb{N}$ with $ng \in N_p \setminus K$. The element $ng + K \cap N$ of the group $(N/K \cap N)_p$ is nonzero, hence $(K/K \cap N)_p = 0$ by 2.2 [1] (it is not $N \subset K$). Consequently, if $x \in K_p$ then $x \in K \cap N$, i.e. $K_p \subset N$. Let $y \in (G_p)^{K_p} \setminus K_p$. Now, there is $m \in \mathbb{N}$ with $my \in N \setminus K$. Since $py \in K_p$, $(p,m) = 1$ and there are integers u,v such that $1 = up + vm$. Hence $y = upy + vmy \in N$. Consequently, $(G_p)^{K_p} \subset N$ for every $p \in \mathbb{P}_K$.

Corollary 2.4. If the intersection of all K-essential subgroups of a group G is zero then $G_t \subset K$.

Theorem 2.5. Let K be a pure subgroup of a group G containing G_t. Then the intersection of all the K-essential torsion-free subgroups is zero.

Proof. Suppose $K \neq G$, otherwise it is trivial. Let $g \in G$ be an element of infinite order and p prime. Let M be a subgroup of G maximal with respect to the properties: $pg \in M$, $g \notin M$, $M_t = 0$. Then M is G_t-essential in G. For, if $x \in G \setminus G_t \cup M$ then either $\langle x,M \rangle_t \neq 0$ or $g \in \langle x,M \rangle$. In the first case, $nx + m = t$; where $n \in \mathbb{N}$, $m \in M$ and $t \in G_t$; hence $\sigma(t)nx \in M \setminus G_t$. In the second case, $g = nx + m$, where $n \in \mathbb{N}$ and $m \in M$; hence $pnx = pg - pm \in M \setminus G_t$. Now, M is K-essential by 3.3 (iii) [1].

The investigation of the intersection of all K-essential subgroups of a group G is connected with the existence-question of the least K-essential subgroup of the group G. If K is a subgroup of a group G then exactly one of the following two cases comes by 1.4 [1]:

(i) There is the least K-essential subgroup N of G. A subgroup M of G is K-essential in G iff $N \subset M$.

(ii) There is no minimal K-essential subgroup of the group G.

Theorem 2.6. Let K be a proper subgroup of a group G. The following are equivalent:

(i) G is torsion;

(ii) A subgroup N of G is K-essential in G iff N contains G_K;

(iii) G_K is the least K-essential subgroup of G.

Proof. (i) $\Longrightarrow$ (iii) If G is torsion then G_K is K-essential in G. For, if $g \in G \setminus K$ then there is $p \in \mathbb{P}_K$ such that we can write $g = a + b$, where $a \in G_p \setminus K_p$ and $b \in \bigoplus_{q \in \mathbb{P}, q \neq p} G_q$.

Let n be the greatest integer such that $p^n a \in G_p \setminus K_p$, i.e. $p^n a \in (G_p)^{K_p}$. If $m = \sigma(b)$ then $mp^n g = mp^n a$. Now, $mp^n a \notin K_p$, since $(m,p) = 1$. Hence $mp^n g \in G_K \setminus K$. The rest follows from 2.3.

(iii) $\Longrightarrow$ (i) See 1.2 [1].

(ii) $\Longleftrightarrow$ (iii) It follows from 1.4 [1].

For example, $\mathbb{Z}(p^{k+1})$ is the least $\mathbb{Z}(p^k)$-essential subgroup of $\mathbb{Z}(p^\infty)$; $\mathbb{Z}(p^{k+1})$ is the least $\mathbb{Z}(p^k)$-essential subgroup of $\mathbb{Z}(p^n)$, where $n > k$.

Theorem 2.7. Let K be a torsion subgroup of a mixed group G. Then the subgroup G_K is the intersection of all K-essential subgroups of G. Moreover, G_K is not K-essential in G, i.e. there is no least K-essential subgroup of G.

<u>Proof</u>. The intersection of all K-essential subgroups of G is torsion by 2.3. On the other hand, the torsion part of the intersection of all K-essential subgroups of G is G_K by 1.8 [1] and 2.6.

<u>Proposition 2.8</u>. Let N and K be subgroups of a group G.

(i) If N is K-essential in G then $N \supset G_K \oplus M$, where M is an essential subgroup of some $(G_K + K)$-high subgroup of G. If K is torsion then the converse holds, too.

(ii) N is G_t-essential in G and torsion-free iff N is essential in some G_t-high subgroup of G.

<u>Proof</u>. (i) If A is a $(G_K + K)$-high subgroup of G then $M = A \cap N$ is essential in A. Now, $N \supset G_K \oplus M$ by 2.3.

Conversely, suppose that K is torsion and $N \supset G_K \oplus M$, where M is an essential subgroup of some $(G_K + K)$-high subgroup A of G. Let $g \in G \setminus K$. If $g \in G_t$ then there is $n \in \mathbb{N}$ such that $ng \in G_K \setminus K$ by 2.6. If $g \notin G_t$ then there is $n \in \mathbb{N}$ such that $ng \in A \oplus (G_K + K)$ and consequently, there is $m \in \mathbb{N}$ with $mng \in M$.

(ii) It follows from (i).

The intersection of all G_t-high subgroups of a group G is zero by Prop. 9 [5]. Now, the intersection of all the G_t-essential torsion-free subgroups is zero by 2.8 (ii). Compare with 2.5.

3. Intersections of maximal K-essential subgroups.

If K is a subgroup of a group G then the maximal subgroups of G that are K-essential in G are called maximal K-essential subgroups of G. The maximal K-essential subgroups of G are exactly maximal elements of the filter of all K-essential subgroups of G.

Definition 3.1. If K is a subgroup of a group G and p is a prime then we denote by K^p the subgroup of G generated by the subgroup pG and by the set of all $x \in G \setminus K$ with $px \in K$.

Lemma 3.2. If K is a subgroup of a group G and p is a prime then

(i) K^p is the least K-essential subgroup of G containing pG;

(ii) pG is K-essential in G iff $K^p = pG$.

Proof. (i) If $g \in G \setminus K$ then either $pg \in pG \setminus K \subset K^p \setminus K$ or $pg \in K$, i.e. $g \in K^p \setminus K$. Consequently, K^p is K-essential in G. Suppose N is K-essential in G containing pG. If $x \notin K$ and $px \in K$ then there is $n \in \mathbb{N}$ with $nx \in N \setminus K$. Now, $(p,n) = 1$ and there are integers u,v such that $1 = up + vn$. Hence $x = upx + vnx \in N$ and $K^p \subset N$.

(ii) It follows from (i).

Theorem 3.3. If K is a subgroup of a group G then the group $\bigcap_{p \in \mathbb{P}} K^p$ is the intersection of all maximal K-essential subgroups of G.

Proof. If M is a maximal subgroup of G then $G/M \cong \mathbb{Z}(p)$ for some prime p; hence $pG \subset M$. Moreover, if M is K-essential in G then $K^p \subset M$ by 3.2. Let $x \notin K^p$ and N be a subgroup of G maximal with respect to the properties: $K^p \subset N$ and $x \notin N$. If

$g \in G \setminus N$ then $x \in \langle g, N \rangle$,i.e. $x = kg + n$, where $n \in N$ and k is an integer. Hence $kg \in \langle x, N \rangle$. Now, $(p,k) = 1$ and there are integers u, v such that $1 = up + vk$. Consequently, $g = upg + vkg \in \langle x, N \rangle$, i.e. $G = \langle x, N \rangle$. Hence N is a maximal subgroup of G. Since $K^p \subset N$, N is K-essential in G. Consequently, K^p is the intersection of all maximal K-essential subgroups of G that contain pG.

Definition 3.4. Let G be a group and K a subgroup of G. An element g of G is said to be K-nongenerator of G if $G = \langle g, M \rangle$, and $\langle M \rangle$ being K-essential in G, imply $G = \langle M \rangle$.

Theorem 3.5. If K is a subgroup of a group G then the intersection of all maximal K-essential subgroups of G is the set of all K-nongenerators of G.

Proof. If $g \in G$ is not a K-nongenerator of G then there is a proper K-essential subgroup N of G such that $G = \langle g, N \rangle$. Denote by M a subgroup of G maximal with respect to the properties: $N \subset M$ and $g \notin M$. The subgroup M is a maximal K-essential subgroup of G and $g \notin M$. Conversely suppose that there is a maximal K-essential subgroup M of G with $g \notin M$. Hence $G = \langle M, g \rangle$ and g is not a K-nongenerator.

Put K = G. It follows from 3.3 that the Frattini subgroup of G is the intersection of all pG with p running over all primes p (see ex. 4, § 3 [3]). By 3.5, the Frattini subgroup of G is the set of all nongenerators of G (see § 62 [6]).

Proposition 3.6. Let K be a subgroup of a free group G. If K is of finite rank then the intersection of all maximal K-essential subgroups of G is zero.

Proof. Let g be a nonzero element of G. By 15.4 [3], we can write $G = \bigoplus_{i=1}^{n} \langle a_i \rangle \oplus G'$, $K = \bigoplus_{i=1}^{n} \langle m_i a_i \rangle$ and $g = \sum_{i=1}^{n} r_i a_i$, where $n \in \mathbb{N}$, m_i are nonnegative integers and r_i are integers, $i = 1,\ldots,n$. Let j be an integer such that $1 \leq j \leq n$ and $r_j \neq 0$; let p be a prime such that $(p,r_j) = 1$ and $(p,m_i) = 1$ for every $i = 1,\ldots,n$. The group pG is K-essential in G. For, if $x \in G \setminus K$, where $x = \sum_{i=1}^{n} s_i a_i + x'$, $s_i \in \mathbb{Z}$, $x' \in G'$, then $px \in pG$. If $px \in K$ then $x' = 0$ and $m_i \mid ps_i$ for each $i = 1,\ldots,n$. Now, $m_i \mid s_i$ for every $i = 1,\ldots,n$ and hence $x \in K$, a contradiction. Since $g \notin pG$, g is not contained in the intersection of all maximal K-essential subgroups of G by 3.2 and 3.3.

From 3.6, it follows that the pure-assumption of the subgroup K of G in 2.5 is not necessary.

4. $\mathcal{N}$-closures and essential topologies. Let G be a group. Let $\mathcal{T}$ be the set of all subgroups T of G such that G/T is a torsion group, and $\mathcal{F}$ be the set of all subgroups F of G such that G/F is torsion-free. Consequently, $\mathcal{T}$ is the set of all G_t-essential subgroups of G (see 3.3 [1]) and $\mathcal{F}$ is the set of all pure subgroups of G containing G_t. The set $\mathcal{T}$ is a filter (see 1.4 [1]) and the set $\mathcal{F}$ is closed under intersections and chain-unions.

For any two subgroups A and B of G define $A \ominus B$ if A is B-essential in G. For a nonempty family $\mathcal{N}$ of subgroups of G put $\mathcal{N}\ominus = \{B;\ A \ominus B \quad \forall A \in \mathcal{N}\}$, $\ominus\mathcal{N} = \{A;\ A \ominus B \quad \forall B \in \mathcal{N}$.

Now, using 1.4, 1.5, 3.3 [1], it follows that the set $\ominus\mathcal{N}$ is a filter. If $\mathcal{N} = \{G\}$ then $\ominus\mathcal{N}$ is the set of all subgroups of G. Otherwise $\ominus\mathcal{N}$ is a subfilter of the filter $\mathcal{T}$ and $\ominus\mathcal{N} = \mathcal{T}$ iff $\mathcal{N} \subset \mathcal{F}$.

The set $\mathcal{N}\ominus$ is closed under intersections and chain-unions and it contains both the largest and the least elements. Denote this least element by $\mathcal{K}(\mathcal{N})$, or $\mathcal{K}(N)$, if $\mathcal{N} = \{N\}$. $\mathcal{K}(\mathcal{N}) = \bigcap \mathcal{N}\ominus$. On the other hand $\mathcal{N} \subset \mathcal{T}$ implies $\mathcal{F} \subset \mathcal{N}\ominus$. If $\mathcal{N} = \{G\}$ then $\mathcal{N}\ominus$ is the set of all subgroups of G; $\mathcal{N} = \mathcal{T}$ implies $\mathcal{N}\ominus = \mathcal{F}$.

<u>Definition 4.1</u>. Let G be a group, $\mathcal{N}$ be a nonempty family of subgroups of G and E be a subset of G. Then the intersection of all subgroups $K \in \mathcal{N}\ominus$ with $E \subset K$ is called $\mathcal{N}$-closure of E and denoted by $\mathcal{N}(E)$. The intersection of all pure subgroups of G containing the group $\langle E, G_t \rangle$ is denoted by $\langle E \rangle_*$.

Obviously, $\langle E \rangle_*$ is a pure subgroup of G for every subset E of G. If $N \in \mathcal{N}$, then $\mathcal{N}(N) = G$.

<u>Theorem 4.2</u>. Let G be a group, $\mathcal{N}$ be a nonempty family of subgroups of G and E be a subset of G. Then

(i) The map $E \longmapsto \mathcal{N}(E)$ is an algebraic closure operator;

(ii) If $\mathcal{N} \not\subset \mathcal{T}$ then $\mathcal{N}(E) = G$;

(iii) If $\mathcal{N} \subset \mathcal{T}$ then $\langle E \rangle \subset \mathcal{N}(E) \subset \langle E \rangle_*$;

(iv) If $\mathcal{N} = \{G\}$ then $\mathcal{N}(E) = \langle E \rangle$;

(v) If $\mathcal{N} = \mathcal{T}$ then $\mathcal{N}(E) = \langle E \rangle_*$.

<u>Proof</u>. Since $\mathcal{N}\ominus$ is closed under intersections and chain-unions, the operator $\mathcal{N}(-)$ is an algebraic closure

operator by II.1.2 [2]. The rest follows from the remarks at the beginning of this section.

Theorem 4.3. Let $\mathcal{N}$ be a nonempty family of subgroups of a group G. If $\mathcal{N} \subset \mathcal{T}$ then $\mathcal{K}(\mathcal{N}) = \bigoplus_{p \in \mathbb{R}} G_p$, where $\mathbb{R}$ is the set of all primes p with $G[p] \not\subset \bigcap \mathcal{N}$. If $\mathcal{N} \not\subset \mathcal{T}$ then $\mathcal{K}(\mathcal{N}) = G$.

Proof. The group $K = \mathcal{K}(\mathcal{N})$ is the intersection of all subgroups L of G, such that each $N \in \mathcal{N}$ is L-essential in G. Let $\mathcal{N} \subset \mathcal{T}$. Denote by $\mathbb{R}$ the set of all primes p with $G[p] \not\subset \bigcap \mathcal{N}$ and $H = \bigoplus_{p \in \mathbb{R}} G_p$. If $K_p \neq G_p$ then $G[p] \subset (G_p)^{K_p} \subset$ $\subset N$ for every $N \in \mathcal{N}$ (by 2.3). Consequently, if $p \in \mathbb{R}$ then $K_p = G_p$ and hence $H \subset K$. For the rest it is sufficient to show that every $N \in \mathcal{N}$ is H-essential in G. Let $g \in G \setminus H$ and $N \in \mathcal{N}$. If g is of infinite order then there is $n \in \mathbb{N}$ with $ng \in N \setminus H$, since G/N is torsion. If g is of finite order then $\sigma(g) =$ $= qr$, where $q \in \mathbb{P} \setminus \mathbb{R}$ and $r \in \mathbb{N}$. Now, $rg \in G[q] \subset N$ and $rg \notin H$. The case $\mathcal{N} \not\subset \mathcal{T}$ is trivial.

Definition 4.4. Let $\mathcal{N}$ be a nonempty family of subgroups of a group G. The topology of G, that is determined by the filter $\ominus\mathcal{N}$ as a base of open neighborhoods about 0, is said to be the $\mathcal{N}$-topology of G, or K-topology of G, if $\mathcal{N} = \{K\}$. $\mathcal{N}$-topologies, with $\mathcal{N}$ running over all nonempty families of subgroups of G, are called the essential topologies of G.

Theorem 4.5. Let G be a group. Then

(i) G-topology of G is discrete. If $\mathcal{N} \neq \{G\}$ then the $\mathcal{N}$-topology of G is nondiscrete;

(ii) If G is not torsion then G_t-topology of G is the finest nondiscrete essentail topology of G. It is Hausdorff and it is identical with each $\mathcal{N}$-topology, where $\{G\} \neq \mathcal{N} \subset \mathcal{F}$;

(iii) If $\mathcal{N}$-topology of G is Hausdorff then $G_t \subset K$ for every $K \in \mathcal{N}$;

(iv) If K is a proper sungroup of G then K_t-topology of G is finer than K-topology of G.

Proof. It follows from 3.3 [1], 2.4 and 2.5.

Corollary 4.6. Torsion groups are exactly the groups with no nondiscrete Hausdorff essential topology. Torsion-free groups are exactly the groups with Hausdorff O-topology.

Remark 4.7. Denote by $\mathcal{A}$ the class of all groups with Hausdorff K-topology, for any subgroup K. Then

(i) $\mathcal{A}$ is closed under subgroups;

(ii) Every free group of finite rank is contained in $\mathcal{A}$;

(iii) Every group from $\mathcal{A}$ is torsion-free.

Proposition 4.8. Let K and L be subgroups of a torsion group G. Then the K-topology of G is finer than the L-topology of G iff $G_K \subset G_L$.

Proof. It follows from 2.6.

Corollary 4.9. The K-topology and the L-topology of a torsion group G are identical iff

(i) $K_p = G_p$ iff $L_p = G_p$,

(ii) $(G_p)^{K_p} = (G_p)^{L_p}$ for every prime p.

Proposition 4.10. Let k and m be nonnegative integers.

Then

(i) The $m\mathbb{Z}$-topology of the group $\mathbb{Z}$ is finer than the $k\mathbb{Z}$-topology of $\mathbb{Z}$ iff $1 \le h_p^{\mathbb{Z}}(m)$ implies $1 \le h_p^{\mathbb{Z}}(k) \le$ $\le h_p^{\mathbb{Z}}(m)$;

(ii) The $m\mathbb{Z}$-topology and $k\mathbb{Z}$-topology of the group $\mathbb{Z}$ are identical iff $m = k$.

Proof. (i) If the $m\mathbb{Z}$-topology is not finer than the $k\mathbb{Z}$-topology then there is a subgroup $n\mathbb{Z}$ of $\mathbb{Z}$, that is $k\mathbb{Z}$-essential in $\mathbb{Z}$ and is not $m\mathbb{Z}$-essential in $\mathbb{Z}$. By 1.10 [1], there is a prime p such that $h_p^{\mathbb{Z}}(n) \ge h_p^{\mathbb{Z}}(m) \ge 1$ and either $h_p^{\mathbb{Z}}(k) = 0$ or $h_p^{\mathbb{Z}}(n) < h_p^{\mathbb{Z}}(k)$.

Conversely, if $h_p^{\mathbb{Z}}(m) = i \ge 1$ and $h_p^{\mathbb{Z}}(k) = 0$ then the subgroup $p^i\mathbb{Z}$ is $k\mathbb{Z}$-essential in $\mathbb{Z}$ and is not $m\mathbb{Z}$-essential in $\mathbb{Z}$ by 1.10 [1]. In case that $h_p^{\mathbb{Z}}(m) = i \ge 1$ and $h_p^{\mathbb{Z}}(k) > h_p^{\mathbb{Z}}(m)$ holds the same.

(ii) It follows from (i).

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