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FINE TOPOLOGIES AS EXAMPLES OF NON-BLUMBERG BAIRE SPACES

Jaroslav LUKEŠ and Luděk ZAJÍČEK, Praha

Abstract: Any $\mathfrak{P}$-harmonic space with countable base in axiomatic potential theory in which the points are polar endowed with the fine topology is non-Blumberg Baire space provided the continuum hypothesis is assumed.

Key words: Blumberg space, Baire space, fine topology in potential theory, density topology.

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In 1922, H. Blumberg [2] showed that for any real function f defined on the real line R, there is a dense subset D of R such that the restriction of f to D is continuous. We shall say that a topological space X is a Blumberg space if for any real function f on X, there is a dense subset D of X such that f/D is continuous. The result of J.C. Bradford and C. Goffman 1960 [3] shows that for a metric space, X is Blumberg if and only if X is a Baire space. While any topological Blumberg space is Baire, the converse is not true in general. The first examples of non-Blumberg Baire space are due to Jr. H.E. White 1974 [9] (assuming the continuum hypothesis, the density topology on the real line serves an example) and 1975 [10] (e.g., any Baire space of cardinality, weight and density character $2^{\aleph_0}$ satisfying the countable chain condition, in which sets of the first category and nowhere dense sets coincide), to R. Levy 1973 [6] (any η_1-set of cardinality $2^{\aleph_0}$) and 1974 [7], and to W.A.R. Weiss 1975 [8] (even an example of compact non-Blumberg space). See also [1], where more detail discussions and interesting results can be found.

Using certain elementary theorem, we will give further examples of non-Blumberg Baire spaces. In particular, any abstract harmonic space equipped with the fine topology sets such an example.

<u>Notation</u>. Given any topological space, $b(A)$ will denote the derived set of A.

<u>Theorem 1</u>. Let X be a topological space without isolated points such that any dense subset of X is of cardinality $2^{\aleph_0}$. If the cardinality of the system $\{b(A);\ A \subset X\}$ is less or equal to $2^{\aleph_0}$, then X is not a Blumberg space.

<u>Proof</u>. For any dense subset A of X, and for any real function f on X we put

$$\tilde{f}_A(y) = \lim_{x \to y, x \neq y, x \in A} \sup f(x) = \sup \{a;\ y \in b\{x \in A;\ f(x) \geqq a\}\},$$
$$y \in X.$$

Since we always have

$$\{y \in X;\ \tilde{f}_A(y) \geqq a\} = \bigcap_{\substack{r<a \\ r \text{ rational}}} b\{y \in A;\ f(y) \geqq r\},$$

it follows that any $\tilde{f}_A$ is measurable with respect to certain system of sets of cardinality $\leqq 2^{\aleph_0}$. By this observation one reaches the conclusion that the system

$$\Phi := \{\tilde{f}_A;\ A \text{ is dense in } X,\ f \text{ is a function on } X\}$$

is of cardinality $\leq 2^{\aleph_0}$. Let Ω be the first ordinal number of cardinality $2^{\aleph_0}$. Suppose now that $\{x_\alpha\}_{\alpha<\Omega}$ is the set of all points of X, and $\{\varphi_\alpha\}_{\alpha<\omega}$ $(\omega \leq \Omega)$ is the set of all functions from Φ. By transfinite induction we can construct a function f on X such that

$$f(x_\alpha) \neq \varphi_\gamma(x_\alpha) \text{ for any } \gamma < \alpha, \ (\gamma < \omega).$$

Then, for any $g \in \Phi$, the cardinality of $\{x \in X;\ f(x) = g(x)\}$ is less than $2^{\aleph_0}$. Hence, it follows easily that there is no dense subset A of X for which f/A is continuous. If it existed, so $\tilde{f}_A \in \Phi$, and this is a contradiction since $f = \tilde{f}_A$ on A and cardinality of A is $2^{\aleph_0}$.

Fine topologies in potential theory. Assume that X is a $\mathfrak{P}$-harmonic-spaoe with countable base in the sense of axiomatics C.Constantinescu and A.Cornea [4]. By this we mean a locally compact topological space with countable base (therefore, X is a metric separable space) which is endowed with a hyperharmonic sheaf and satisfies certain axioms. The fine topology on X is the coarsest topology on X which is finer than the initital topology and in which any hyperharmonic function on X is continuous. It is known that there are not isolated points in the fine topology ([4], Corollary 5.1.2), and that X endowed with the fine topology is a Baire space ([4], Corollary 5.1.1). Moreover, if we shall suppose that the points of X are polar, then the derived set b(A) of any subset $A \subset X$ in the fine topology is exactly the set of all points of X where A is not thin ([4], Exercise 7.2.1). Therefore, b(A) is always of type G_δ in the initial topology ([4], Corollary 7.2.1), and thus the cardinality of the system $\{b(A);\ A \subset X\}$ is less or equal to $2^{\aleph_0}$. Further, the whole space X is uncountable ([4], Exercise 5.1.5), and any countable subset of X is polar. Hence, it is always closed in the fine topology. Thus, assuming the continuum hypothesis, any dense subset of X must be of cardinality $2^{\aleph_0}$.

Applying our theorem, we get the following important examples of non-Blumberg Baire spaces.

Theorem 2. Assuming the continuum hypothesis, any abstract $\mathfrak{B}$-harmonic space with a countable base endowed with the fine topology, in which every point is polar, is a non-Blumberg Baire space.

Remark. The same theorem remains true if we suppose that the points of X are semi-polar only and axiom of thinness (= any semi-polar set is finely closed) is satisfied. In both cases, we can also replace the continuum hypothesis with the assumption that any subset of X of cardinality $< 2^{\aleph_0}$ is semi-polar. (It is sufficient to use the facts that, in the fine topology, any semi-polar set is of the first category and the whole space X is of the second category in itself.)

Density topology. Consider now the ordinary density topology in the Euclidean space R^n introduced by C. Goffman and D. Waterman 1961 in [5]. In this topology R^n is a Baire space without isolated points. Moreover, any derived set in density topology is of type $G_{\delta\sigma}$ in the Euclidean topology.

Thus, the theorem 1 gives again the following result which is due to Jr. H.E. White.

<u>Theorem 3</u>. If any subset of R^n of cardinality $< 2^{\aleph_0}$ has a Lebesgue measure zero, then R^n endowed with the density topology is a Baire non-Blumberg space.

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Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

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