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VAN DER WAERDEN THEOREM FOR SEQUENCES OF INTEGERS NOT CONTAINING AN ARITHMETIC PROGRESSION OF k TERMS

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Abstract: A theorem stated in the title is proved by a direct construction.

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Introduction. As analogy to [3] and [1] it was conjectured by P. Erdős the following (see [0]): For every k, r there exists a set of integers N not containing an arithmetic progression of r + 1 terms with the property that for every partition of the set N into k classes there exists an arithmetic progression with r terms in one of the classes. The purpose of this note is to prove this theorem. In fact we prove here a stronger theorem ("the prototype theorem" in [4]) from which one can deduce the characterization theorem for partition properties of classes of sets if integers which do not contain "long" arithmetic progressions.

After this paper was written we were informed that J. Spencer in about the same time solved independently the Erdős's problem. Meanwhile the Spencer's solution was published in [7]. His method uses strongly a theorem of Hales-

Jewett [8]. Our proof is by a direct construction and as it gives a slightly stronger result we decided to publish it anyway.

Results: For natural numbers $a, b, a \le b$ put $[a,b] = \{a, a+1, \dots, b\}$. Let $M = \{m_0, \dots, m_r\}$, $N = \{n_0, \dots, n_s\}$ be sets of natural numbers (these sets will be always considered with the relativized ordering of $\mathbb{N}$ and the notation will be always chosen with respect to this ordering; i.e. we assume $m_0 < m_1 < \dots < m_r$).

A mapping $f: M \longrightarrow N$ is said to be sequential iff there exists a positive constant d such that $f(m_i) = f(m_0) + d(m_i - m_0)$ and $m_i = m_0 + a \in M \Longleftrightarrow f(m_0) + d\, a \in N$.

The van der Waerden theorem [6] then states that for every k,r there exists a finite set of natural numbers N such that for every mapping $c: N \longrightarrow [1,k]$ there exists a sequential mapping $f: [1,r] \longrightarrow N$ such that $c \circ f$ is a constant mapping (we write $c \circ f = \S$ if the actual value of the constant is of no importance).

Denote by Seq the class of all finite subsets of $\mathbb{N}$ and by Seq (r), $r \ge 2$, the class of all finite subsets of $\mathbb{N}$ which do not contain an arithmetic progression with r + 1 terms (equivalently $M \in$ Seq $(r) \Longleftrightarrow$ there exists no sequential mapping $f: [1, r+1] \longrightarrow M$).

We prove:

Theorem 1: Let $r \ge 2$, $k \ge 1$ be fixed. For every $M \in$ Seq (r) then there exists a set $N \in$ Seq (r) such that for every mapping $c: N \longrightarrow [1,k]$ there exists a sequential mapping $f: M \longrightarrow N$ such that $c \circ f = \S$.

Clearly this theorem implies:

Corollary: Let $r \geq 2$ be fixed. Then the class Seq (r) with sequential mappings) has A-partition property $\Longleftrightarrow |A| = 1$. (See [4,5] for the definition of A-partition property.) To see this, one has only to observe that for every $r \geq 2$ one can colour by two colours all arithmetic progressions with r terms in $\mathbb{N}$ in such a way that each arithmetic progression with r + 1 terms contains arithmetic progressions of both colours. (This is well known.) Thus we have the perfect analogy with the situation in graphs: the characterization theorem of partition properties of classes Seq (r), compare [4].

The proof of the theorem 1 is a convenient modification of the Graham-Rothschild proof of van der Waerden theorem [2]. We introduce now parameters and on each step of the induction procedure we check that the resulting set belongs to Seq (r).

Proofs: We write shortly (x_i) for $(x_i; i \in [1,m])$ if there is no danger of confusion.

Let r,m be positive integers, $\emptyset \neq \omega \subseteq \varkappa \in$ Seq (r), moreover, let ω and $\varkappa$ satisfy: $x \in \omega$, $y < x$, $y \in \varkappa \Longrightarrow y \in \omega$. Denote by $S(\omega, \varkappa, r, m)$ the following statement:

For every positive integer k there exists a set $N = N(\omega, \varkappa, r, m, k)$ with the following properties: 1) $N \in \mathrm{Seq}(r)$; 2) For every mapping $c: N \longrightarrow [1,k]$ there are numbers $a, d_1, d_2, \dots, d_m$ such that

A1: $c(a + \sum_{i=1}^{m} x_i d_i) = c(a + \sum_{i=1}^{m} y_i d_i)$ whenever

$(x_i) \in \omega^m$, $(y_i) \in \omega^m$;

A2: $(x_i) \in \varkappa^m \Longleftrightarrow a + \sum_{i=1}^{m} x_i d_i \in N.$

We prove

<u>Theorem 2</u>: The statement $S(\omega, \varkappa, r, m)$ is valid for each admissible choice of parameters.

<u>Proof</u>: The proof will be by induction on $|\omega|$ and m (for each admissible choice of $\varkappa$, and r).

Clearly $S(\omega, \varkappa, r, 1)$, $|\omega| = 1$, is always valid.

The induction step will follow from two claims:

<u>Claim 1</u>: Let $S(\omega, \varkappa, r, m')$ be valid for each $m' \leq m$. Then there holds $S(\omega, \varkappa, r, m + 1)$.

<u>Proof</u>: Let k be fixed. Let $N_1 = N(\omega, \varkappa, r, m, k)$, $|N_1| = a$ and $N_2 = N(\omega, \varkappa, r, 1, k^a) = \{n_i;\ i \in [1,b]\}$ be fixed (both sets exist by induction hypothesis). Define N by $N = \bigcup \{N_1 + (n_i - n_1)D;\ i \in [1,b]\}$ where $D = ar$ and $N_1 + (n_i - n_1)D = \{n + (n_i - n_1)D;\ n \in N_1\}$.

We prove $N = N(\omega, \varkappa, r, m + 1, k)$.

1) Assume $N \notin \mathrm{Seq}\,(r)$: let $P = \{a + jd;\ j \in [c, r]\}$ be an arithmetic progression in N. Then either there exists $i \in [1,b]$ such that $|P \cap (N_1 + (n_i - n_1)D)| \geq 2$ and in this case $P \subseteq N_1 + (n_i - n_1)D$ (by the choice of D) which is a contradiction with the properties of N_1 or $|P \cap (N_1 + (n_i - n_1)D)| \leq 1$ for each $i \in [1,b]$ and in this case we get a contradiction with the properties of N_2.

2) Let $c: N \longrightarrow [1,k]$ be a fixed mapping. We have an induced mapping $c': N_2 \longrightarrow [1,k]^{N_1}$ defined by $c'(i) = c\,|_{N_1 + (n_i - n_1)D}$. By the properties of N_2 there are n_A, D_1 such that

1) $c'(n_A + iD_1) = \S$ for all $i \in \omega$;

2) $n_A + iD_1 \in N_2 \Longleftrightarrow i \in \varkappa$.

Furthermore (by the properties of N_1) there are $a, d_1, d_2, \ldots$ $\ldots, d_m$ such that

1) $c(a + \sum_{i=1}^{m} x_i d_i + (n_A + xD_1 - n_1)D) =$
$= c(a + \sum_{i=1}^{m} x_i' d_i + (n_A + x'D_1 - n_1)D)$ whenever $(x_i) \in \omega^m$, $(x_i') \in \omega^m$, $x \in \omega$, $x' \in \omega$,

2) $a + \sum_{i=1}^{m} x_i d_i + (n_A + xD_1 - n_1)D \in N \Longleftrightarrow (x_i) \in \varkappa^m$, $x \in \varkappa$.

Put $\bar{a} = a + (n_A - n_1)D$, $\bar{d}_i = d_i$ for $i \in [1,m]$, $\bar{d}_{m+1} = D_1 D$. For these parameters the statements A1 and A2 are valid.

Claim 2: Let $S(\omega, \varkappa, r, m)$ be valid for each m. Assume $\omega \neq \varkappa$, let $q \in \varkappa \setminus \omega$ be the minimal number, put $\bar{\omega} =$ $= \omega \cup \{q\}$. Then there holds $S(\bar{\omega}, \varkappa, r, 1)$.

Proof: Let k be a fixed positive integer. Take $N =$ $N(\omega, \varkappa, r, k, k) \in \mathrm{Seq}\ (r)$. Let $c: N \longrightarrow [1,k]$ be a fixed mapping. By the properties of N there are $a, d_1, \ldots, d_k$ such that

1) $c(a + \sum_{i=1}^{k} x_i d_i) = c(a + \sum_{i=1}^{k} y_i d_i)$ whenever $(x_i) \in \omega^k$, $(y_i) \in \omega^k$

2) $a + \sum_{i=1}^{k} x_i d_i \in N \Longleftrightarrow (x_i) \in \omega^k$.

Consider the numbers a, $a + qd_1, \ldots, a + \sum_{i=1}^{k} qd_i \in N$. Using Dirichlets principle there are $0 \leq u < v \leq k$ such that $c(a + \sum_{i=1}^{u} qd_i) = c(a + \sum_{i=1}^{v} qd_i)$. But then $f(x) = a + \sum_{i=1}^{u} qd_i + x \sum_{i=u+1}^{v} d_i$ for $x \in \bar{\omega}$ is a desirable sequential mapping $\bar{\omega} \longrightarrow N$ with the property $c \circ f = \S$.

Moreover, $a + \sum_{i=1}^{u} qd_i + x \sum_{i=u+1}^{v} d_i \in N \Longleftrightarrow x \in \mathfrak{X}$.

This finishes the proof of Claim 2 and of Theorem 2.

Now the theorem 1 is equivalent to the statement $S(M,M,r,1)$, $M \in Seq(r)$. Let us state explicitly:

Corollary: For every r and k positive integers there exists a set N of natural numbers such that:

1) N does not contain an arithmetic progression with r + 1 terms;

2) for every partition of N into k classes there exists an arithmetic progression with r terms in one of the classes.

Remark: Given r, the bound given by the above proof on the size of the set $N([1,r], [1,r], r, k)$ is extremely large. However as the above proof is closely related to the proof of van der Waerden theorem we obtain similar bounds for these two theorems. This is one of the indications of weakness of the proof of van der Waerden theorem.

References

[0] P. ERDOS: Problems and results in combinatorial number theory, Société Mathématique de France, Astérisque 24-25(1975), 295-310.

[1] J. FOLKMAN: Graphs with monochromatic complete subgraphs in every edge colouring, SIAM J. Applied Math. 18(1970), 19-29.

[2] R.L. GRAHAM, B.L. ROTHSCHILD: A short proof of van der Waerden's theorem on arithmetic progressions, Proc. Amer. Math. Soc. 42(1974), 385-386.

[3] J. NEŠETŘIL, V. RÖDL: Ramsey property of graphs with

forbidden complete subgraphs, J. Comb. Th. (B), 20,3(1976), 243-249.

[4] J. NEŠETŘIL, V. RÖDL: Type theory of partition problems of graphs, in: Recent advances in graph theory, Academia, Prague (1975), 405-412.

[5] J. NEŠETŘIL, V. RÖDL: Partitions of finite relational and set systems, to appear in J. Comb.Th.(A).

[6] B.L. van der WAERDEN: Beweis einer Baudetschen Vermutung, Nieuw Arch. Wis. 15(1928), 212-216.

[7] J. SPENCER: Restricted Ramsey configurations, J. Comb. Th. 19,3(1975), 278-286.

[8] A. HALES, R.I. JEWETT: Regularity and positional games, Trans. Amer. Math. Soc. 106(1963), 222-229.

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