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ON COLLECTIVE COMPACTNESS OF DERIVATIVES

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Abstract: F being a family of mappings in locally convex spaces and F' being the family of their derivatives, the necessary and sufficient conditions for F under which F' is collectively precompact, are given.

Key words: Locally convex space, Gâteaux and Fréchet derivatives, strong equicontinuity, collective precompactness, Orlicz space.

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The concept of collective compactness is a natural generalization of the notion of compactness for single mappings. It was introduced by Anselone and Moore in [4] and then studied in detail by various authors in [1] - [3],[5] - [9], [11], [12]. A great deal of those papers is devoted to the collective compactness of a family of linear operators in Banach spaces because of its important applications in the theory of approximate solutions of operator equations. Nevertheless, the more general relations, concerning the concept of collective compactness, were also studied. For instance, Lloyd investigated in [11] the connections between collective precompactness of a family of nonlinear mappings in topological linear spaces and collective precompactness of the family of derivatives of those mappings.

The aim of our paper is to establish an analogy of the well-known theorem of Palmer [13] (on complete continuity of the derivative of a mapping in Banach spaces) for families of mappings in topological linear spaces. That means, we will find necessary and sufficient conditions for a family of nonlinear mappings under which the family of derivatives of those mappings will be collectively precompact. Thus, some of our results complete partly the results of Lloyd [11]; namely, this concerns our Theorem 2.2 generalizing Theorem 3.11 of [11]. Our main results are presented in Theorems 2.4, 4.4, 4.5 and 5.1.

1. Notations and definitions. Throughout the paper, X and Y will denote arbitrary locally convex topological linear spaces over the real field R, $\mathcal{U}$ and $\mathcal{V}$ will denote the collections of all neighbourhoods of 0 in X and Y, respectively, $\mathcal{U}_0$ and $\mathcal{V}_0$ will denote the collections of all open convex balanced neighbourhoods of 0 in X and Y, respectively and X^* and Y^* will denote the topological duals of X and Y. M will denote an arbitrary open convex subset of X and $\mathcal{B}$ and $\mathcal{B}_M$ will denote the collections of all bounded subsets of X and M, respectively. We will denote by $\mathcal{L}(X,Y)$ the space of all continuous linear mappings from X into Y with the topology of uniform convergence on bounded subsets of X, and by $\mathfrak{X}$ the base of neighbourhoods of 0 in $\mathcal{L}(X,Y)$ consisting of all sets of the type $(B,V) = \{u \in \mathcal{L}(X,Y) : u(B) \subset V\}$ where $B \in \mathcal{B}$ and $V \in \mathcal{V}$.

Let $\mathcal{F}$ be a family of mappings from M into Y. This family is said to be weakly (resp., strongly) equicontinuous [11] on M iff for each $x_0 \in M$ and each bounded net $(x_\lambda : \lambda \in L) \subset M$ (i.e., L is a directed set and the set $\{x_\lambda : \lambda \in L\}$ is bounded), weak convergence $x \rightharpoonup x_0$ implies $f(x) \rightharpoonup f(x_0)$ (resp., $f(x) \to f(x_0)$) uniformly over $f \in \mathcal{F}$. The family $\mathcal{F}$ is said to be uniformly weakly (resp., strongly) equicontinuous on $N \subset M$ iff for any bounded nets $(x_\lambda : \lambda \in L)$, $(x'_\lambda : \lambda \in L) \subset N$, weak convergence $x_\lambda - x'_\lambda \rightharpoonup 0$ implies $f(x_\lambda) - f(x'_\lambda) \rightharpoonup 0$ (resp., $f(x_\lambda) - f(x'_\lambda) \to 0$) uniformly over $f \in \mathcal{F}$.

The family $\mathcal{F}$ is said to be collectively precompact [11] on M iff for each $B \in \mathcal{B}_M$, the set $\{f(x): x \in B, f \in \mathcal{F}\}$ is precompact in Y. (Recall that precompactness is equivalent to relative compactness in complete spaces.) Similarly, the family $\mathcal{F}'$ of derivatives f' (see below) of mappings from $\mathcal{F}$ is collectively precompact on M iff for each $B \in \mathcal{B}_M$, the set $\{f'(x): x \in B, f \in \mathcal{F}\}$ is precompact in $\mathcal{L}(X,Y)$. The family $\mathcal{F}'$ of derivatives is said to be collectively jointly precompact [11] on M iff for each $B_1 \in \mathcal{B}_M$ and $B_2 \in \mathcal{B}$, the set $\{f'(x)h: x \in B_1, h \in B_2, f \in \mathcal{F}\}$ is precompact in Y.

We use the following concept of differentiability which is due to Averbukh and Smolyanov [15],[16] (see also [11]). A mapping $f: M \to Y$ is said to be Gâteaux (resp., Fréchet) differentiable at $x \in M$, iff there exists $u \in \mathcal{L}(X,Y)$ such that for each $h \in X$ (resp., $B \in \mathcal{B}$) and $V \in \mathcal{V}$, there exists $\delta > 0$ such that

$$f(x + th) - f(x) - u(th) \in tV$$

whenever $|t| \leq \delta$ (resp., whenever $h \in B$ and $|t| \leq \delta$); such a mapping u is denoted by $f'(x)$. A mapping $f: M \longrightarrow Y$ is said to be Gâteaux (resp., Fréchet) differentiable on $N \subset M$ iff it is Gâteaux (resp., Fréchet) differentiable at every $x \in N$. A mapping $f: M \longrightarrow Y$ is said to be uniform differentiable on $N \subset M$ iff it is Fréchet differentiable at every $x \in N$, and, given $B \in \mathcal{B}$ and $V \in \mathcal{V}$, the $\delta > 0$ in the definition above can be chosen independently of $x \in N$.

For a differentiable mapping $f: M \longrightarrow Y$, the notation

$$\omega_f(x,h,t) = f(x + th) - f(x) - f'(x)(th)$$

($x \in M$, $h \in X$, $t \in R$) will be used throughout this paper.

A family $\mathcal{F}$ of mappings from M into Y is said to be Gâteaux (resp., Fréchet) equidifferentiable at $x \in M$, iff each $f \in \mathcal{F}$ is Gâteaux (resp., Fréchet) differentiable at x, and given $h \in X$ (resp., $B \in \mathcal{B}$) and $V \in \mathcal{V}$, the $\delta > 0$ in the definition above can be chosen independently of $f \in \mathcal{F}$. The equidifferentiability and the uniform equidifferentiability of $\mathcal{F}$ on $N \subset M$ is defined in an evident way.

Throughout the paper, for a given family $\mathcal{F}$ of mappings, the following notations are used for point sets and families of mappings induced by $\mathcal{F}$:

$\mathcal{F}(x) = \{f(x): f \in \mathcal{F}\}$, $\mathcal{F}' = \{f': f \in \mathcal{F}\}$,

$\mathcal{F}'(x) = \{f'(x): f \in \mathcal{F}\}$, $\mathcal{F}(N) = \{f(x): x \in N, f \in \mathcal{F}\}$, and similar ones.

We remark that it must be distinguished between precompactness of $\mathcal{F}'(x)$ (as a subset of $\mathcal{L}(X,Y)$) and collective precompactness of $\mathcal{F}'(x)$ (as a family of mappings from X into Y); see [1] for detail discussion in that direction (for instance, both concepts are equivalent for compact self-adjoint operators in a Hilbert space).

2. Necessary conditions. Throughout this section, $\mathcal{F}$ will denote a family of Gâteaux differentiable mappings from $M \subset X$ into Y.

Our first assertion (and its proof, too) is a slight modification of ([11], Theorem 3.8).

Theorem 2.1. Let the family $\mathcal{F}'$ be collectively precompact on M. Then the family $\mathcal{F}$ is weakly equicontinuous on M uniformly on each bounded subset of M.

Proof. Suppose $\mathcal{F}$ is not uniformly weakly equicontinuous on a set $N \in \mathfrak{B}_M$. Then there exist nets $(x_\lambda : \lambda \in L)$, $(x'_\lambda : \lambda \in L) \subset N$, $(f_\lambda : \lambda \in L) \subset \mathcal{F}$, a continuous linear functional $e^* \in Y^*$ and $\varepsilon > 0$ such that $x'_\lambda - x_\lambda \longrightarrow 0$ $(\lambda \in L)$ and that

(1) $$|\langle f_\lambda(x'_\lambda) - f_\lambda(x_\lambda), e^*\rangle| > \varepsilon$$

for all $\lambda \in L$. According to the well-known mean value theorem, for every $\lambda \in L$, there exists $t_\lambda \in (0,1)$ such that

$$\langle f_\lambda(x'_\lambda) - f_\lambda(x_\lambda), e^*\rangle = \langle f'_\lambda(x_\lambda + t_\lambda(x'_\lambda - x_\lambda)) \cdot$$
$$\cdot (x'_\lambda - x_\lambda), e^*\rangle .$$

Set $z_\lambda = f'_\lambda(x_\lambda + t_\lambda(x'_\lambda - x_\lambda))$ and let $V \in \mathcal{V}$ be such that $|\langle y, e^* \rangle| \leq \frac{1}{2}\varepsilon$ whenever $y \in V$. Denoting by B the balance hull of the set $\{x'_\lambda - x_\lambda : \lambda \in L\}$, the set $\mathcal{F}'((N + B) \cap M)$ is precompact in $\mathcal{L}(X,Y)$ and so we can choose a subnet $(z_\lambda : \lambda \in L')$ of $(z_\lambda : \lambda \in L)$ such that $z_{\lambda_1} - z_{\lambda_2} \in (B,V)$ for each $\lambda_1, \lambda_2 \in L'$. Similarly as in the proof of ([11], Th. 3.8), we can now prove that $|\langle f_\lambda(x'_\lambda) - f_\lambda(x_\lambda), e^* \rangle| \leq \varepsilon$ for each $\lambda \in L'$, which contradicts (1).

The following theorem improves the result of ([11], Th. 3.11).

<u>Theorem 2.2</u>. Let the family $\mathcal{F}'$ be collectively jointly precompact on M and let the set $\mathcal{F}(x_0)$ be precompact in Y for some $x_0 \in M$. Then the family $\mathcal{F}$ is collectively precompact on M.

<u>Proof</u>. Suppose there exists $N \in \mathcal{B}_M$ such that $\mathcal{F}(N)$ is not precompact; that means there are nets $(f_\lambda : \lambda \in L) \subset \mathcal{F}$, $(x_\lambda : \lambda \in L) \subset N$ and a neighbourhood $V \in \mathcal{V}$ such that

$$(2) \qquad f_{\lambda_1}(x_{\lambda_1}) - f_{\lambda_2}(x_{\lambda_2}) \notin V$$

for all $\lambda_1, \lambda_2 \in L$. Let $W \in \mathcal{V}_0$ be such that $4W \subset V$.

According to the mean value theorem ([11], Th.1.6),

$$f_\lambda(x_\lambda) - f_\lambda(x_0) \in \overline{\mathrm{co}}\{f'_\lambda(x_0 + t(x_\lambda - x_0))(x_\lambda - x_0) : t \in [0,1]\}$$

for each $\lambda \in L$, where $\overline{\mathrm{co}}$ denotes the closed convex

hull. Hence, for every $\lambda \in L$, there is $\{a_\lambda^t : t \in [0,1]\} \subset$
$\subset [0,1]$ and $r_\lambda \in W$ such that

$$(3) \quad f_\lambda(x_\lambda) = f_\lambda(x_0) + \sum_{t \in [0,1]} a_\lambda^t f'_\lambda(x_0 + t(x_\lambda - x_0)) \cdot$$
$$\cdot (x_\lambda - x_0) + r_\lambda \quad ,$$

where $\sum_{t \in [0,1]} a_\lambda^t = 1$ and only a finite number of a_λ^t (for each fixed λ) is non-zero.

Denote by B the balance hull of $\{x_\lambda - x_0 : \lambda \in L\}$; it is $B \in \mathcal{B}$ and $x_0 + t(x_\lambda - x_0) \in (x_0 + B) \cap M \in \mathcal{B}_M$ for all $\lambda \in L$ and $t \in [0,1]$. Hence, denoting

$$y_\lambda^t = f'_\lambda(x_0 + t(x_\lambda - x_0))(x_\lambda - x_0) \; ,$$

the set $\{y_\lambda^t : \lambda \in L, t \in [0,1]\}$ is precompact, and its convex hull C is then precompact by the well-known theorem, too. The net $(\sum_{t \in [0,1]} a_\lambda^t y_\lambda^t : \lambda \in L)$ lies in C and so there exists some Cauchy subnet $(\sum_{t \in [0,1]} a_\lambda^t y_\lambda^t : \lambda \in L')$. Hence, there is $\lambda' \in L'$ such that

$$(4) \qquad \sum_{t \in [0,1]} a_{\lambda_1}^t y_{\lambda_1}^t - \sum_{t \in [0,1]} a_{\lambda_2}^t y_{\lambda_2}^t \in W$$

for all $\lambda_1, \lambda_2 \in L'$ whenever $\lambda_1, \lambda_2 \vdash \lambda'$.

By the assumption of our theorem, we can choose a Cauchy subnet $(f_\lambda(x_0) : \lambda \in L'')$ of the net $(f_\lambda(x_0) : \lambda \in L')$ and hence, there is $\lambda'' \in L''$ such that $\lambda'' \vdash \lambda'$ and

$$(5) \qquad f_{\lambda_1}(x_0) - f_{\lambda_2}(x_0) \in W$$

for all $\lambda_1, \lambda_2 \in L''$, $\lambda_1, \lambda_2 \succ \lambda''$. It follows now from (3),(4),(5) that for all $\lambda_1, \lambda_2 \in L''$,

$$f_{\lambda_1}(x_{\lambda_1}) - f_{\lambda_2}(x_{\lambda_2}) = [f_{\lambda_1}(x_0) - f_{\lambda_2}(x_0)] + r_{\lambda_1} - \\ - r_{\lambda_2} + [\sum_{t \in [0,1]} a^t_{\lambda_1} y^t_{\lambda_1} - \sum_{t \in [0,1]} a^t_{\lambda_2} y^t_{\lambda_2}] \in V$$

whenever $\lambda_1, \lambda_2 \succ \lambda''$, which contradicts (2).

The following theorem extends the well-known result concerning a single mapping (see e.g. [14]), to the case of a family of mappings.

Theorem 2.3. Let $\mathcal{F}$ be weakly equicontinuous on M (resp., uniformly weakly equicontinuous on each bounded subset of M). If $\mathcal{F}$ is collectively precompact on M then it is strongly equicontinuous on M (resp., uniformly strongly equicontinuous on each bounded subset of M).

Proof. Suppose $\mathcal{F}$ is uniformly weakly equicontinuous on each bounded subset of M and is not uniformly strongly equicontinuous on some $N \in \mathcal{B}_M$; then there are nets $(x_\lambda : \lambda \in L)$, $(x'_\lambda : \lambda \in L) \subset N$, $(f_\lambda : \lambda \in L) \subset \mathcal{F}$ and $V \in \mathcal{V}$ such that $x_\lambda - x'_\lambda \longrightarrow 0$ $(\lambda \in L)$ and

(6) $$f_\lambda(x_\lambda) - f_\lambda(x'_\lambda) \notin V$$

for every $\lambda \in L$. Moreover,

(7) $$\langle f(x_\lambda) - f(x'_\lambda), e^* \rangle \longrightarrow 0 \qquad (\lambda \in L)$$

holds for all $f \in \mathcal{F}$ and $e^* \in Y^*$.

Choose arbitrary Cauchy subnets $(f_\lambda(x_\lambda) : \lambda \in L')$ and $(f_\lambda(x'_\lambda) : \lambda \in L')$ of nets $(f_\lambda(x_\lambda) : \lambda \in L)$ and

$(f_\lambda(x'_\lambda) : \lambda \in L)$, respectively. Denoting by $\widetilde{Y}$ the completion of Y (such a complete Hausdorff space that Y is dense in $\widetilde{Y}$ and the topology of Y induced from $\widetilde{Y}$ is equivalent to the original one; $\widetilde{Y}$ is also locally convex), there are $y_0, y'_0 \in \widetilde{Y}$ such that

$$(8) \qquad f_\lambda(x_\lambda) \longrightarrow y_0 \, , \quad f_\lambda(x'_\lambda) \longrightarrow y'_0 \qquad (\lambda \in L')$$

in the topology of $\widetilde{Y}$. Hence,

$$(9) \quad \langle f_\lambda(x_\lambda) - f_\lambda(x'_\lambda), \widetilde{e}^* \rangle \longrightarrow \langle y_0 - y_0, \widetilde{e}^* \rangle \qquad (\lambda \in L')$$

for every $\widetilde{e}^* \in \widetilde{Y}^*$.

Since the restriction of an arbitrary $\widetilde{e}^* \in \widetilde{Y}^*$ is an element of Y^* , it follows from (7) that

$$(10) \quad \langle f_\lambda(x_\lambda) - f_\lambda(x'_\lambda), \widetilde{e}^* \rangle \longrightarrow 0 \qquad (\lambda \in L') \, .$$

Thus, we obtain from (9) and (10) that $\langle y_0 - y'_0, \widetilde{e}^* \rangle = 0$ for every $\widetilde{e}^* \in \widetilde{Y}^*$ and so $y_0 - y'_0 = 0_{\widetilde{Y}} = 0_Y$. Whence by (8),

$$f_\lambda(x_\lambda) - f_\lambda(x'_\lambda) \longrightarrow 0 \qquad (\lambda \in L')$$

in the topology of $\widetilde{Y}$ and hence in that of Y , too.

This contradicts (6) and so proves the "uniform" part of our theorem; the "simple" part can be proved in a similar way.

Now, we are ready to present the main result of this section:

Theorem 2.4. Let the family $\mathcal{F}'$ be collectively precompact on M, the mapping $f'(x)$ be precompact for every $f \in \mathcal{F}$ and $x \in M$ and let the set $\mathcal{F}(x_0)$ be precompact for some $x_0 \in M$. Then the family $\mathcal{F}$ is collectively precompact on M and uniformly strongly equicontinuous on each bounded subset of M.

Proof. By ([11], Th. 3.10), the family $\mathcal{F}'$ is collectively jointly precompact on M. Hence, the result follows from Theorems 2.1, 2.2 and 2.3.

3. The property $\mathbf{C}$. Throughout this section, $\mathcal{F}$ will be an arbitrary family of mappings from M into Y.

Definition 3.1. The family $\mathcal{F}$ possesses the property $\mathbf{C}$ at some point $x_0 \in M$ iff the following condition holds: For every net $(f_\lambda : \lambda \in L) \subset \mathcal{F}$, a subnet $(f_\lambda : \lambda \in L')$ can be chosen in such manner that, given arbitrary $B \in \mathcal{B}$ and $V \in \mathcal{V}$, there exist $r_{BV} > 0$ such that $x_0 + r_{BV} B \subset M$ and for every δ, $0 < \delta \leq r_{BV}$, there is $\lambda_\delta \in L'$ such that

$$(11) \qquad f_{\lambda_1}(x_0 + h) - f_{\lambda_2}(x_0 + h) \in \delta V$$

for each $\lambda_1, \lambda_2 \in L'$, $\lambda_1, \lambda_2 \succ \lambda_\delta$ and each $h \in \delta B$.

It is evident that if $\mathcal{F}$ possesses the property $\mathbf{C}$ at $x_0 \in M$ then the set $\mathcal{F}(x_0)$ is precompact in Y. Two following theorems will make the meaning of the property $\mathbf{C}$ more clear.

Theorem 3.1. Let $x_0 \in M$ and suppose that every net in $\mathcal{F}$ contains a subnet that is uniformly Cauchy on some neighbourhood of x_0, i.e. there exists $U \in \mathcal{U}$ such that for every given $V \in \mathcal{V}$, an index λ_V can be chosen such that $f_{\lambda_1}(x) - f_{\lambda_2}(x) \in V$ holds for all $\lambda_1, \lambda_2 \succ \lambda_V$ and all $x \in x_0 + U$. Then $\mathcal{F}$ possesses the property $\mathbb{C}$ at x_0 .

<u>Proof</u>. Let the condition of Theorem 3.1 hold and let $(f_\lambda : \lambda \in L) \subset \mathcal{F}$ be an arbitrary net. Choose a subnet $(f_\lambda : \lambda \in L')$ and $U \in \mathcal{U}$ as described in the theorem above and such that $x_0 + U \subset M$. Let $B \in \mathcal{B}$ and $V \in \mathcal{V}$ be arbitrary and choose $r_{BV} > 0$ so that $r_{BV} B \subset U$. Given any σ, $0 < \sigma \leq r_{BV}$, the formula (11) evidently holds for all $h \in \sigma B$ and all $\lambda_1, \lambda_2 \in L'$, $\lambda_1, \lambda_2 \succ \lambda_{V'}$, where $V' = \sigma V$.

<u>Remark</u>. The condition of Theorem 3.1 implies $\mathcal{F}$ is collectively precompact on some neighbourhood of x_0 .

<u>Proposition 3.1</u>. Let $\mathcal{F}$ be Fréchet equidifferentiable at a point $x_0 \in M$. Then $\mathcal{F}$ possesses the property $\mathbb{C}$ at x_0 if and only if both sets $\mathcal{F}(x_0)$ and $\mathcal{F}'(x_0)$ are precompact in Y and $\mathcal{L}(X,Y)$, respectively.

<u>Proof</u>. 1) Suppose $\mathcal{F}$ is equidifferentiable and possesses the property $\mathbb{C}$ at x_0 . Precompactness of $\mathcal{F}(x_0)$ in Y is a direct consequence of Definition 3.1 and so it remains to prove precompactness of $\mathcal{F}'(x_0)$ only.

Let $(f'_\lambda(x_0) : \lambda \in L)$ be an arbitrary net, $(f_\lambda :$ $: \lambda \in L)$ be the corresponding net in $\mathcal{F}$ and let $(f_\lambda :$ $: \lambda \in L')$ be its subnet chosen according to Definition 3.1. We will prove that the corresponding subnet $(f'_\lambda(x_0) : \lambda \in$ $\in L')$ of $(f'_\lambda(x_0) : \lambda \in L)$ is Cauchy in $\mathcal{L}(X,Y)$.

Let an arbitrary $(B,V) \in \mathfrak{X}$ be given, let $W \in \mathcal{V}_0$ be such that $4\,W \subset V$. There exists σ_0 such that $0 < \sigma_0 \leq r_{BW}$ (the number from Definition 3.1) and that

$$\omega f_{\lambda}(x_0,h,t) \in tW$$

whenever $|t| \leq \sigma_0$, $h \in B$ and $\lambda \in L'$. By the definition of the property $\mathbb{C}$ again, there is $\lambda_{\sigma_0} \in L'$ such that for every $\lambda_1, \lambda_2 \in L'$, $\lambda_1, \lambda_2 \succ \lambda_{\sigma_0}$ and each $h \in B$, it holds

$$f'_{\lambda_1}(x_0)h - f'_{\lambda_2}(x_0)h = \frac{1}{\sigma_0}[f_{\lambda_1}(x_0 + \sigma_0 h) - f_{\lambda_2}(x_0 + \sigma_0 h) + f_{\lambda_2}(x_0) - f_{\lambda_1}(x_0) + \omega_{f_{\lambda_1}}(x_0,h,\sigma_0) - \omega_{f_{\lambda_2}}(x_0,h,\sigma_0)] \in \frac{1}{\sigma_0}[\sigma_0 W + \sigma_0 W + \sigma_0 W + \sigma_0 W] \subset V,$$

whence $f'_{\lambda_1}(x_0) - f'_{\lambda_2}(x_0) \in (B,V)$ follows.

2) Suppose $\mathcal{F}$ is equidifferentiable at x_0 and the sets $\mathcal{F}(x_0)$ and $\mathcal{F}'(x_0)$ are precompact; we will prove that $\mathcal{F}$ has the property $\mathbb{C}$ at x_0.

For an arbitrary net $(f_\lambda : \lambda \in L)$ in $\mathcal{F}$, there exists a subnet $(f_\lambda : \lambda \in L')$ such that the corresponding nets $(f_\lambda(x_0) : \lambda \in L')$ and $(f'_\lambda(x_0) : \lambda \in L')$ are Cauchy nets.

Let $B \in \mathcal{B}$ and $V \in \mathcal{V}$ be arbitrary and let $W \in \mathcal{V}_0$ be such that $4\,W \subset V$. Choose $r_{BW} \in (0,1)$ so that $x_0 + r_{BW}B \subset M$ and

$$\omega_{f_\chi}(x_0,h,t) \in tW$$

whenever $|t| \leq r_{BV}$, $h \in B$ and $\lambda \in L'$.

Given any δ, $0 < \delta \leq r_{BV}$, there is $\lambda_\delta \in L'$ such that

$$f_{\lambda_1}(x_0) - f_{\lambda_2}(x_0) \in \delta W$$

$$f'_{\lambda_1}(x_0) - f'_{\lambda_2}(x_0) \in (B,W)$$

for every $\lambda_1, \lambda_2 \in L'$, $\lambda_1, \lambda_2 \succ \lambda_\delta$. It follows now that for all such λ_1, λ_2 and all $h \in \delta B$,

$$f_{\lambda_1}(x_0 + h) - f_{\lambda_2}(x_0 + h) = f_{\lambda_1}(x_0 + \delta k) -$$
$$- f_{\lambda_2}(x_0 + \delta k) = [f_{\lambda_1}(x_0) - f_{\lambda_2}(x_0)] +$$
$$+ [f'_{\lambda_1}(x_0) - f'_{\lambda_2}(x_0)] (\delta k) + \omega_{f_{\lambda_1}}(x_0, k, \delta) -$$
$$- \omega_{f_{\lambda_2}}(x_0, k, \delta) \in 4\delta W \subset \delta V$$

(where $k = \frac{1}{\delta} h \in B$) holds.

Fréchet equidifferentiability of $\mathcal{F}$ at x_0 and boundedness of the set $\mathcal{F}'(x_0)$ in $\mathcal{L}(X,Y)$ imply equicontinuity of $\mathcal{F}$ at x_0 if the space X is bornological (see [1] and [15]). Hence, the following consequence of our proposition holds:

<u>Corollary 3.1</u>. Let X be bornological. Let $\mathcal{F}$ be Fréchet equidifferentiable at $x_0 \in M$ and possess the property $\mathbb{C}$ at x_0. Then $\mathcal{F}$ is equicontinuous at x_0.

4. __Sufficient conditions.__ In the following theorem, $\mathcal{F}$ is assumed to be a family of mappings from the closure $\overline{M}$ of M in X into Y. We suppose the space X is semireflexive, which means that each bounded subset of X is relatively weakly compact; if X is barrelled then semireflexivity is equivalent to reflexivity of X.

__Theorem 4.1.__ Let X be semireflexive, $\mathcal{F}$ be strongly equicontinuous on $\overline{M}$ and let the set $\mathcal{F}(x)$ be precompact for each $x \in M_0$ where M_0 is dense in M. Then $\mathcal{F}$ is collectively precompact on $\overline{M}$.

__Proof.__ Suppose there exist $N \in \mathcal{B}_{\overline{M}}$ such that $\mathcal{F}(N)$ is not precompact, i.e. there are nets $(f_\lambda : \lambda \in L) \subset \mathcal{F}$ and $(x_\lambda : \lambda \in L) \subset N$ and $V \in \mathcal{V}$ such that

$$f_{\lambda_1}(x_{\lambda_1}) - f_{\lambda_2}(x_{\lambda_2}) \notin V \tag{12}$$

for every $\lambda_1, \lambda_2 \in L$. Let $W \in \mathcal{V}_0$ be such that $5W \subset V$.

Being bounded in X, the set $\{x_\lambda : \lambda \in L\}$ is relatively weakly compact. Being closed and convex, the set $\overline{M}$ is weakly closed. Hence, there exists a subnet $(x_\lambda : \lambda \in L')$ of $(x_\lambda : \lambda \in L)$ and $x_0 \in \overline{M}$ so that $x_\lambda \longrightarrow x_0$ $(\lambda \in L')$, which implies $f(x_\lambda) \longrightarrow f(x_0)$ $(\lambda \in L')$ uniformly over $f \in \mathcal{F}$.

Choose $\lambda_0 \in L'$ so that

$$f(x_\lambda) \in f(x_0) + W \tag{13}$$

for all $\lambda \in L'$, $\lambda \succ \lambda_0$ and all $f \in \mathcal{F}$. Since $\mathcal{F}$ is equicontinuous on $\overline{M}$ and M_0 is dense in $\overline{M}$, there is $x_0' \in M_0$ so that

(14) $$f(x_0') \in f(x_0) + W$$

for every $f \in \mathcal{F}$. Finally, there exists a subnet $(f_\lambda : \lambda \in L'')$ of $(f_\lambda : \lambda \in L)$ such that

(15) $$f_{\lambda_1}(x_0') - f_{\lambda_2}(x_0') \in W$$

for all $\lambda_1, \lambda_2 \in L''$. It follows now from (13),(14),(15) that for all $\lambda_1, \lambda_2 \in L''$, $\lambda_1, \lambda_2 \vdash \lambda_0$,

$$f_{\lambda_1}(x_{\lambda_1}) - f_{\lambda_2}(x_{\lambda_2}) \in V ,$$

which contradicts (12).

Hereafter, we shall suppose $\mathcal{F}$ is a family of mappings that are defined on some neighbourhood M^+ of M in X and are Gâteaux differentiable on M ; we can suppose that $M^+ = \bar{M} + U_0$ where $U_0 \in \mathcal{U}_0$.

Applying Theorem 4.1 to the family $\mathcal{F}'$ instead of $\mathcal{F}$ and using Proposition 3.1, the following result can be obtained.

Corollary 4.1. Let X be semireflexive and suppose $\mathcal{F}$ is equidifferentiable and possesses the property $\mathbb{C}$ at each point of some set M_0 dense in M . If $\mathcal{F}'$ is strongly equicontinuous on M then it is collectively precompact on M .

Theorem 4.2. Let X be semireflexive and let $\mathcal{F}$ be strongly equicontinuous on M^+ and uniformly equidifferentiable on each bounded subset of $\bar{M}$. Then $\mathcal{F}'$ is strongly equicontinuous on $\bar{M}$.

Proof. Suppose the conditions of Theorem 4.2 hold but

$\mathcal{F}'$ is not strongly equicontinuous on $\overline{M}$. Then there exist nets $(x_\lambda : \lambda \in L) \subset M$ and $(f_\lambda : \lambda \in L) \subset \mathcal{F}$, $x_0 \in \overline{M}$ and $Z \in \mathcal{Z}$ such that $x_\lambda \longrightarrow x_0$ $(\lambda \in L)$, $\{x_\lambda : \lambda \in L\}$ is bounded and

$$f'_\lambda(x_\lambda) \notin f'_\lambda(x_0) + Z$$

for all $\lambda \in L$. Let $Z = (B,V)$ where $B \in \mathcal{B}$ and $V \in \mathcal{V}$; then, for every $\lambda \in L$, there is $h_\lambda \in B$ such that

$$f'_\lambda(x_\lambda)h_\lambda \notin f'_\lambda(x_0)h_\lambda + V. \tag{16}$$

The set $\{h_\lambda : \lambda \in L\}$ is bounded, hence it is relatively weakly compact and hence, there is a subnet $(h_\lambda : \lambda \in L')$ of $(h_\lambda : \lambda \in L)$ and $h_0 \in X$ such that $h_\lambda \longrightarrow h_0$ $(\lambda \in L')$.

Denote $B_0 = B \cup \{h_0\}$ and let $W \in \mathcal{V}_0$ be such that $4W \subset V$. There exists $\delta \in (0,1)$ so that $\delta B_0 \subset U_0$ (see the definition of M^+) and that

$$\omega_f(x_\lambda, h_\lambda, t) \in tW, \quad \omega_f(x_0, h_\lambda, t) \in tW \tag{17}$$

for each $f \in \mathcal{F}$, $\lambda \in L'$ and $|t| \leq \delta$.

Let $\lambda_0 \in L'$ be such that

$$f(x_\lambda) - f(x_0) \in \delta W \tag{18}$$

$$f(x_\lambda + \delta h_\lambda) - f(x_0 + \delta h_\lambda) \in \delta W$$

for all $\lambda \in L'$, $\lambda \succ \lambda_0$ and all $f \in \mathcal{F}$. It follows now from (17) and (18) that

$$f'_\lambda(x_\lambda)h_\lambda - f'_\lambda(x_0)h_\lambda = \frac{1}{\delta}[f_\lambda(x_\lambda + \delta h_\lambda) - f_\lambda(x_0 + \delta h_\lambda) +$$

$$+ f_\lambda(x_0) - f_\lambda(x_\lambda) + \omega_{f_\lambda}(x_0, h_\lambda, \delta) - \omega_{f_\lambda}(x_\lambda, h_\lambda, \delta)] \in V$$

whenever $\lambda \in L'$, $\lambda \succ \lambda_0$. Thus, we have a contradiction to (16).

Theorem 4.3. Let X be bornological. Let $\mathcal{F}$ be equicontinuous on M and Fréchet equidifferentiable at some point $x_0 \in M$. Then the family $\mathcal{F}'(x_0)$ is equicontinuous on X.

Proof. Let arbitrary $h_0 \in X$ and $V \in \mathcal{V}$ be given, let $W \in \mathcal{V}_0$ be such that $2W \subset V$.

Select an arbitrary $B \in \mathcal{B}_M$ and let $\delta \in (0,1)$ be such that $x_0 + \delta B \subset M$ and $\omega_f(x_0, h, t) \in tW$ whenever $|t| \leq \delta$, $h \in B$ and $f \in \mathcal{F}$. There exists $U \in \mathcal{U}$ so that $f(x_0 + U) \subset f(x_0) + W$ for all $f \in \mathcal{F}$. Let $\delta_0 \in (0, \delta)$ be so that $\delta_0 B \subset U$. Then, for every $h \in B$, $|t| \leq \delta_0$ and $f \in \mathcal{F}$, it follows that

$$f'(x_0)th = f(x_0 + th) - f(x_0) - \omega_f(x_0, h, t) \in W - tW \subset V .$$

Put $U_0 = \bigcap_{f \in \mathcal{F}} [f'(x_0)]^{-1}(V)$. We have just proved that U_0 absorbs B; hence, since B was arbitrary, it follows U_0 is a neighbourhood of 0.

Equicontinuity of $\mathcal{F}'(x_0)$ at h_0 is proved.

Now, the main result of this section can be established:

Theorem 4.4. Let X be semireflexive. Suppose a family $\mathcal{F}$ is strongly equicontinuous on M^+, uniformly equidifferentiable on each bounded subset of $\bar{M}$ and possesses the property $\mathbf{C}$ at each point of M_0 where M_0 is a dense subset of $\bar{M}$. Then both families $\mathcal{F}$ and $\mathcal{F}'$ are collectively precompact on $\overline{M}$ and the family $\mathcal{F}'(x)$ is collectively precompact on X for every $x \in \overline{M}$.

Proof. Collective precompactness of $\mathcal{F}$ follows immediately from Theorem 4.1 and Proposition 3.1, collective precompactness of $\mathcal{F}'$ follows from Corollary 4.1 and Theorem 4.2. The result concerning $\mathcal{F}'(x)$ follows from collective precompactness and equidifferentiability of $\mathcal{F}$ (see [11], Th. 3.9).

Remark. It follows from Theorem 4.2 that under the assumptions of Theorem 4.4, the family $\mathcal{F}'$ is strongly equicontinuous on $\overline{M}$. If the space X is bornological, the family $\mathcal{F}'(x)$ is equicontinuous on X for every $x \in \overline{M}$ according to Theorem 4.3.

We terminate this section by the following slight modification of Theorem 2.4 to show the close relation between our sufficient condition for collective precompactness of $\mathcal{F}'$ (Theorem 4.4) and the necessary one. In fact, Theorem 4.5 below is nearly a converse to Theorem 4.4.

Let $\mathcal{F}$ be as in Section 2. The assertions of the following theorem immediately follow from Theorem 2.4 and Proposition 3.1.

Theorem 4.5. Let $\mathcal{F}$ be Fréchet equidifferentiable on a set $M_0 \subset M$. Suppose the set $\mathcal{F}(x_0)$ is precompact in Y for some $x_0 \in M$, the family $\mathcal{F}'$ is collectively precompact on M and $f'(x)$ is precompact on X for each $f \in \mathcal{F}$ and $x \in M$. Then the family $\mathcal{F}$ is collectively precompact and strongly equicontinuous on M uniformly on each bounded subset of M and possesses the property $\mathbf{C}$ at each point of M_0 .

5. Some particular cases. First, we examine the case of $M = X$. In this case, Theorem 4.5 is precisely a converse to Theorem 4.4 and hence, the following equivalence holds:

Theorem 5.1. Let X, Y be Hausdorff locally convex spaces, X be semireflexive, and let $\mathcal{F}$ be a family of mappings from X into Y . Suppose the family $\mathcal{F}$ is uniformly equidifferentiable on each bounded convex subset of X . Then $\mathcal{F}$ is strongly equicontinuous and possesses the property $\mathcal{C}$ on X if and only if the family $\mathcal{F}'$ is collectively precompact on X , all mappings $f'(x)$ $(f \in \mathcal{F}$, $x \in M)$ are precompact and the set $\mathcal{F}(x_0)$ is precompact in Y for some $x_0 \in X$.

Remark. It follows from the theorems of the preceding section that in the theorem above, the statement " $\mathcal{F}'$ is collectively precompact" can be equivalently replaced by " $\mathcal{F}'$ is strongly equicontinuous".

In the second part of this section, we will investigate the case of normed linear spaces. In such case, the following property can be introduced:

Definition 5.1. Let X, Y be normed linear spaces, $\mathcal{F}$ be a family of mappings from $M \subset X$ into Y , $x_0 \in M$. The family $\mathcal{F}$ is said to possess the property $\mathcal{C}_0$ at x_0 iff the following condition holds: Given any sequence $\{f_n\} \subset \mathcal{F}$, a subsequence $\{f_{n_k}\}$ can be chosen such that, for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that any δ , $0 < \delta \leq$ $\leq r_\varepsilon$, being given, the inequality

$$\| f_{n_k}(x_0 + h) - f_{m_k}(x_0 + h) \| \leq \varepsilon \sigma$$

holds for every $h \in X$, $\| h \| = \sigma$, and all sufficiently large n_k, m_k.

Theorem 5.2. Let $X, Y, \mathcal{F}, x_0$ be as in the definition above. Suppose that for every sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ that is uniformly Cauchy on each sufficiently small sphere with a centre at x_0, i.e. a number $r_0 > 0$ can be given such that for every $\varepsilon > 0$ and r, $0 < r \leq r_0$, there is $n_{\varepsilon r}$ such that

$$\| f_{n_k}(x) - f_{m_k}(x) \| \leq \varepsilon$$

for all $x \in M$, $\| x - x_0 \| = r$, whenever $n_k, m_k \geq n_{\varepsilon r}$. Then $\mathcal{F}$ possesses the property $\mathfrak{C}_0$ at x_0.

Proof of this theorem is trivial and can be omitted. We remark that in contrast to the condition of Theorem 3.1, the condition of Theorem 5.2 does not imply collective precompactness of $\mathcal{F}$ on a neighbourhood of x_0. Moreover, in contrast to the property $\mathfrak{C}$, the property $\mathfrak{C}_0$ at x_0 does not imply precompactness of $\mathcal{F}(x_0)$ in Y; nevertheless, the following assertion holds:

Lemma 5.1. Let $X, Y, \mathcal{F}, x_0$ be as above. If $\mathcal{F}$ is equicontinuous at x_0 and possesses the property $\mathfrak{C}_0$ at that point, then the set $\mathcal{F}(x_0)$ is precompact in Y.

Proof. Let $\{f_n(x_0)\}$ be an arbitrary sequence of points from $\mathcal{F}(x_0)$ and denote by $\{f_{n_k}(x_0)\}$ its subsequence defined by the property $\mathfrak{C}_0$. We will prove that $\{f_{n_k}(x_0)\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be an arbitrary number and choose $r_0 > 0$ so that $x_0 + h \in M$ and

$$(19) \qquad \| f(x_0) - f(x_0 + h) \| \leq \frac{\varepsilon}{4}$$

for all $f \in \mathcal{F}$ and $h \in X$ whenever $\| h \| \leq r_0$. Set $\sigma = \min(1, r_0, r_{\frac{\varepsilon}{2}})$ where $r_{\frac{\varepsilon}{2}} > 0$ is the number defined by the property $\mathcal{C}_0$. Now, it follows by the property $\mathcal{C}_0$ that there is $n_0 = n_{\frac{\varepsilon}{2}, \sigma}$ such that

$$(20) \qquad \| f_{n_k}(x_0 + h) - f_{m_k}(x_0 + h) \| \leq \frac{\varepsilon}{2} \sigma \leq \frac{\varepsilon}{2}$$

for all $n_k, m_k \geq n_0$ and $h \in X$, $\| h \| = \sigma$. Choosing an arbitrary $h_0 \in X$, $\| h_0 \| = \sigma$, it follows from (19) and (20) that

$$\| f_{n_k}(x_0) - f_{m_k}(x_0) \| \leq \| f_{n_k}(x_0) - f_{n_k}(x_0 + h_0) \| +$$
$$+ \| f_{n_k}(x_0 + h_0) - f_{m_k}(x_0 + h_0) \| + \| f_{m_k}(x_0 + h_0) -$$
$$- f_{m_k}(x_0) \| \leq \varepsilon$$

whenever $n_k, m_k \geq n_0$, and this completes the proof.

Using the lemma above, the following assertion can be proved in a similar way as Proposition 3.1.

<u>Proposition 5.1</u>. Let X, Y, $\mathcal{F}$, x_0 be as above and let $\mathcal{F}$ be Fréchet equidifferentiable at x_0. Then $\mathcal{F}$ is equicontinuous at x_0 and possesses the property $\mathcal{C}_0$ at x_0 if and only if the sets $\mathcal{F}(x_0)$ and $\mathcal{F}'(x_0)$ are precompact in Y and $\mathcal{L}(X,Y)$, respectively.

Corollary 5.1. Under assumptions of Proposition 5.1, the family $\mathfrak{F}$ possesses the property $\mathbb{C}$ at x_0 if and only if it is equicontinuous at x_0 and possesses the property $\mathbb{C}_0$ at that point.

Corollary 5.2. Let X and Y be normed linear spaces. Then Corollary 4.1 and Theorems 4.4, 4.5 and 5.1 will remain true even if we replace everywhere $\mathbb{C}$ by $\mathbb{C}_0$.

Note that in the case when X and Y are Banach spaces, $M = X$ and $\mathfrak{F} = \{f\}$ (i.e., $\mathfrak{F}$ consists of a single mapping), our Theorem 5.1 reduces to the well-known theorem of Palmer [13] on compactness of the derivative of a mapping.

Eventually, we will examine the case of Orlicz spaces (see e.g.[10] for definitions and notations used below). An Orlicz space L^*_Φ is not reflexive in general, however, it follows from ([10], Th. 14.4) that it is always E_Ψ-reflexive, where Ψ is the complementary function to Φ and E_Ψ is the closure of the set of all bounded functions in L^*_Ψ . Thus, all previous assertions will be valid also for arbitrary Orlicz spaces if we write everywhere E_Ψ-weak (resp., E_Ψ-strong) equicontinuity instead of ordinary weak (resp., strong) equicontinuity and others like that.

References

[1] P.M. ANSELONE: Collectively Compact Operator Approximation Theory and Applications to Integral Equations, Prentice-Hall, 1971.

[2] P.M. ANSELONE: Collectively compact and totally bounded sets of linear operators, J.Math.Mech.17 (1968), 613-622.

[3] P.M. ANSELONE: Compactness properties of sets of operators and their adjoints, Math.Z. 113(1970),233-236.

[4] P.M. ANSELONE, R.H. MOORE: Approximate solutions of integral and operator equations, J.Math.Anal.Appl. 9(1964), 268-277.

[5] P.M. ANSELONE; T.W. PALMER: Collectively compact sets of linear operators, Pac.J.Math.25(1968), 417-422.

[6] P.M. ANSELONE, T.W. PALMER: Spectral analysis of collectively compact strongly convergent operator sequences, Pac.J.Math. 25(1968), 423-431.

[7] J.W. DANIEL: Collectively compact sets of gradient mappings, Indag. Math. 30(1968), 270-279.

[8] J.D. De PREE; J.A. HIGGINS: Collectively compact sets of linear operators, Math.Z. 115(1970),366-370.

[9] M.V. DESHPANDE, N.E. JOSHI: Collectively compact and semi-compact sets of linear operators in topological vector spaces, Pac.J.Math. 43(1972),317-326.

[10] M.A. KRASNOSELSKIJ, J.B. RUTICKIJ: Convex Functions and Orlicz Spaces, Moscow, 1958 (Russian).

[11] J. LLOYD: Differentiable mappings on topological vector spaces, Studia Math. 45(1973),147-160 and 49(1973-4), 99-100.

[12] R.H. MOORE: Differentiability and convergence of compact nonlinear operators, J.Math.Anal.Appl. 16(1966), 65-72.

[13] K.J. PALMER: On the complete continuity of differentiable mappings, J.Austr.Math.Soc. 9(1969), 441-444.

[14] M. VAINBERG: Variational Methods for the Study of Nonlinear Operators, Moscow, 1956 (Russian).

[15] V.I. AVERBUKH, O.G. SMOLYANOV: The theory of differentiation in linear topological spaces, Usp. Mat. Nauk 22(1967),201-258 (Russian).

[16] V.I. AVERBUKH, O.G. SMOLYANOV: The various definitions of the derivative in linear topological spaces, Usp.Mat.Nauk 23(1968), 67-113 (Russian).

[17] M.Z. NASHED: Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials ... ; in Nonlinear Functional Analysis and Applications (ed. J. B. Rall), New York 1971.

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