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ON NORMALITY RELATION AND ITS GENERALIZATION ON LATTICES

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Abstract: Normality relation and its generalization are on a lattice L binary, unsymmetric and reflexive relations with restricted substitution properties. The lattices of these relations are considered in the case where L is a finite lattice, and a decomposition theorem is proved.

Key words: Finite lattices, normality relations, generalizations, the lattice of relations, decomposition.

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1. Preliminaries and introduction. A binary relation N on a lattice L is called a normality relation on L, if it satisfies the following conditions of Dean and Kruse (see Beran [1]):

(DK0) aNa for each $a \in L$.

(DK1) $aNb \Longrightarrow a \leq b$.

(DK2) (aNb and cNd) $\Longrightarrow a \wedge c N b \wedge d$.

(DK3) (aNb and aNc) $\Longrightarrow aNb \vee c$.

(DK4) (aNb and cNd) $\Longrightarrow a \vee c N a \vee c \vee (b \wedge d)$.

(DK5) $\{a \leq b$ and $(aNa \vee c$ or $cNa \vee c)\} \Longrightarrow a \vee (b \wedge c) = b \wedge (a \vee c)$.

We shall call a binary relation on L satisfying the conditions (DK0) - (DK3) a generalized normality relation.

As one can easily see, normality and generalized normality relations on a lattice are unsymmetric generalizations of lattice congruences and lattice tolerances (see e.g. Zelinka and Chajda [2]). The purpose of this paper is to determine a few properties of the lattice N(L) of all normality relations and of the lattice GN(L) of all generalized normality relations on a finite lattice L. It will be shown that in a class of finite distributive lattices, a lattice of this class is directly decomposable if and only if there are two non-trivial generalized normality relations GK and GM on L such that $GK \vee GM = 1$ and $GK \wedge GM = 0$ in the lattice GN(L).

The condition (DK5) is a restricted modularity condition, and hence it is valid in each modular lattice.

As a general reference in lattice theory we have used the monograph [4] of G. Szász. The few terms of graph theory of this paper can be found in the book [3] of F. Harary.

2. Joins and meets of relations. At first we give a characterization of normality relations in terms of sublattices of a finite modular lattice.

Let L be a finite lattice. We denote by $\mathcal{A} = \{A_t \mid t \in T\}$ a family of convex sublattices of L, where T is a set of indices, and by 0_t and 1_t the least and greatest elements of A_t, respectively. Further, we assume that for each $x \in L$ there is a sublattice $A_t \in \mathcal{A}$ such that $x = 0_t$.

Theorem 1. Let L be a finite modular lattice. Each family $\mathcal{A}$ of convex sublattices of L determine a normality relation on L and conversely, each N determines such a family if and only if for any two indices $s, u \in T$ there exist indices $p,r \in T$ such that

(i) $0_s \wedge 0_u = 0_p$ and $1_s \wedge 1_u \leq 1_p$,

(ii) $0_s \vee 0_u = 0_r$ and $0_s \vee 0_u \vee (1_s \wedge 1_u) \leq 1_r$.

<u>Proof</u>. 1^o: Let $\mathcal{A}$ be a family with properties given in the theorem. We define a binary antisymmetric relation on L given by $\mathcal{A}$ as follows:

$0_s Rx \Longleftrightarrow x \in A_s \in \mathcal{A}$.

We show that R is a normality relation on L.

aRa for each $a \in L$, as for each $a \in L$ there was a sublattice $A_t \in \mathcal{A}$ such that $0_t = a$, and so (DK0) holds. (DK1) follows directly from the definition of R.

(DK2): Let aRb and cRd. According to the definition $a = 0_s$ and $c = 0_u$ for some indices $u,s \in T$. Further, $a \wedge c = 0_s \wedge 0_u = 0_p$ and $0_p \leq b \wedge d \leq 1_s \wedge 1_u \leq 1_p$ for some $p \in T$, and thus the definition of R implies $0_p R b \wedge d$.

(DK3): Let aRb and aRc, i.e. $a,b,c \in A_t$ for some $t \in T$. As A_t is a sublattice of L, $b \vee c \in A_t$, and so $aRb \vee c$. The proof of (DK4) is similar to that of (DK2), and (DK5) holds, as L is modular.

2^o: Let N be a given normality relation on L. We shall show that N generates a family $\mathcal{F}$ of convex sublattices of L having the same properties as $\mathcal{A}$ in the theorem. Let $F_x = \{y \mid xNy, y \in L\}$ for each $x \in L$, and we denote $\mathcal{F} = \{F_x \mid x \in L\}$.

As xNx holds for each $x \in L$, there is, according to (DK1), for each $x \in L$ a set $F_x \in \mathcal{F}$ such that x is the least element of F_x. As F_x is finite, there exists an element $w = \bigvee\{y \mid y \in F_x\}$, and according to (DK3), xNw. For each

$v \in [x,w] \subseteq L$ it holds vNv. By applying (DK2) to xNw and vNv, we obtain xNv. Hence $F_x = [x,w]$, which is a convex sublattice of L.

Let xNy and zNv. According to (DK2), $x \wedge z N y \wedge v$, and on the other hand $F_{x \wedge y} \in \mathcal{F}$. As $x \wedge z N y \wedge v$, then $y \wedge v \leq 1_{x \wedge z}$, and so (i) holds. (ii) follows similarly from (DK4), and (DK5) holds, as L is modular. This completes the proof.

The following corollary follows immediately from the proof above.

<u>Corollary</u>. Let L be a finite lattice. Each family $\mathcal{A}$ of convex sublattices of L determines a generalized normality relation GN on L and conversely, GN determines such a family if and only if for any two indices $s,u \in T$ there exists an index $p \in T$ such that (i) of Theorem 1 holds.

In the following we look for meets and joins of two generalized normality relations (normality relations). The assertion of the following lemma is obviously valid.

<u>Lemma 1</u>. Let L be a finite lattice and GN and GR two generalized normality relations on L. The relation K, where $aKb \Longleftrightarrow \{aGNb \text{ and } aGRb\}$ is a generalized normality relation on L and $K = GN \wedge GR$.

Analogous lemma holds also for normality relations.

If GM is a generalized normality relation on a finite lattice L we denote the corresponding family of intervals of L by $\mathcal{A}(GM)$, an interval of $\mathcal{A}(GM)$ with the least element $x \in L$ by A_{GMx} and the greatest element of A_{GMx} by 1_{GMx}. The following theorem gives the most simple join of two generalized normality relations.

Theorem 2. Let GM and GN be two generalized normality relations on a finite distributive lattice L. The family $\mathcal{A}$ (GH), where $A_{GMx} = [x, 1_{GMx} \vee 1_{GNx}]$, determines a generalized normality relation on L and GH = GM $\vee$ GN if and only if

(i) $L = L_1 \times L_2 \times \dots \times L_m$, where L_i is a chain, $i = 1,\dots,m$,

or

(ii) L can be divided into two convex sublattices L^* and L^{**} such that $L^* \cap L^{**}$ contains only one element, which is 0 of L^* and 1 of L^{**}, L^{**} is a chain and L^* satisfies the condition (i) above.

Proof. 1^0: Let L satisfy (i) of the theorem; it is sufficient to show the validity of (DK2) - the conditions (DK0), (DK1) and (DK3) hold obviously.

Let aGHb and cGHd; we shall show that $d \wedge b \leq (1_{GMa} \vee 1_{GNa}) \wedge (1_{GMc} \vee 1_{GNc}) \leq 1_{GMa\wedge c} \vee 1_{GNa\wedge c}$. At first, by applying the distributivity, $(1_{GMa} \vee 1_{GNa}) \wedge (1_{GMc} \vee 1_{GNc}) = (1_{GMa} \wedge 1_{GMc}) \vee \vee (1_{GNa} \wedge 1_{GNc}) \vee (1_{GMa} \wedge 1_{GNc}) \vee (1_{GNa} \wedge 1_{GMc})$, where $1_{GMa} \wedge \wedge 1_{GMc} \leq 1_{GMa\wedge c}$ and $1_{GNa} \wedge 1_{GNc} \leq 1_{GNa\wedge c}$, as GM and GN are generalized normality relations on L. In the following we consider the term $1_{GMa} \wedge 1_{GNc}$ and show that it is equal to or less than $1_{GNa\wedge c} \vee 1_{GMa\wedge c}$; the proof is similar for $1_{GMc} \wedge 1_{GNa}$.

As $L = L_1 \times \dots \times L_m$, $a = (a_1, a_2, \dots, a_m)$, $c = (c_1, \dots, c_m)$, $1_{GMa} = (x_1, \dots, x_m)$ and $1_{GNc} = (y_1, \dots, y_m)$, where $a_i, c_i, x_i, y_i \in L_i$. As $aGM1_{GMa}$ and $cGN1_{GNc}$, we obtain $(a_1, \dots, a_i, \dots \dots, a_m) GM (a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_m)$ and $(c_1, \dots, c_i, \dots \dots, c_m) GN (c_1, \dots, c_{i-1}, y_i, c_{i+1}, \dots, c_m)$. Furthermore, as L_i is a chain, $a_i \leq c_i$ or $c_i \leq a_i$, and we assume that

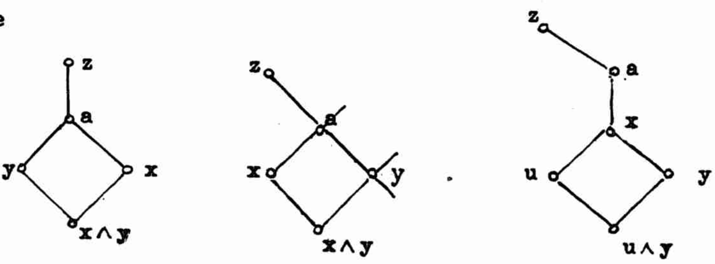
$a_i \leq c_i$, i.e. $a_i \wedge c_i = a_i$, and $x_i \wedge y_i \leq x_i$ holds always. But then $(a_1,\dots,a_m)GM(a_1,\dots,a_{i-1},x_i,a_{i+1},\dots,a_m)$ implies $(a_1,\dots,a_{i-1},a_i \wedge c_i,a_{i+1},\dots,a_m)GM(a_1,\dots,a_{i-1},x_{i-1},x_i \wedge y_i, a_{i+1},\dots,a_m)$. According to the properties (DK0) and (DK2) of GM, we can now form the meet of both sides with $(c_1,\dots ,c_{i-1},y_i,c_{i+1},\dots,c_m)$, and we obtain $(a_1 \wedge c_1,\dots,a_m \wedge c_m)GM(a_1 \wedge c_1,\dots,a_{i-1} \wedge c_{i-1},x_i \wedge y_i,a_{i+1} \wedge c_{i+1},\dots,a_m \wedge c_m)$ as $c_i \leq y_i$. So, in general, for each i, $(a_1 \wedge c_1,\dots,a_m \wedge c_m)GT(a_1 \wedge c_1,\dots,a_{i-1} \wedge c_{i-1},x_i \wedge y_i,a_{i+1} \wedge c_{i+1},\dots,a_m \wedge c_m)$, where GT is GM or GN, $i = 1,\dots,m$. Let z be the join of all elements $(a_1 \wedge c_1,\dots,a_{i-1} \wedge c_{i-1},x_i \wedge y_i, a_{i+1} \wedge c_{i+1},\dots ,a_m \wedge c_m)$ which are in the relation GN with $(a_1 \wedge c_1,\dots ,a_m \wedge c_m)$ for some value of i, and let the corresponding join be w in the case of GM; these joins exist according to (DK3). As GM and GN are generalized normality relations and $a \wedge cGMw$ and $a \wedge cGNz$, $w \leq 1_{GMa\wedge c}$ and $z \leq 1_{GNa\wedge c}$, and trivially, $w \vee z = (x_1 \wedge y_1,\dots,x_m \wedge y_m) = 1_{GMa} \wedge 1_{GNc}$, where $w \vee z \leq 1_{GMa\wedge c} \vee 1_{GNa\wedge c}$. As mentioned above, we can similarly see that $1_{GMc} \wedge 1_{GNa} \leq 1_{GMa\wedge c} \vee 1_{GNa\wedge c}$.

As each term of the join $(1_{GMa} \wedge 1_{GMc}) \vee (1_{GNa} \wedge 1_{GNc}) \vee (1_{GMa} \wedge 1_{GNc}) \vee (1_{GMc} \wedge 1_{GNa})$ is less or equal to $1_{GMa\wedge c} \vee 1_{GNa\wedge c}$, the join satisfies this relation as well. Hence $(1_{GMa} \vee 1_{GNa}) \wedge (1_{GMc} \wedge 1_{GNc}) \leq 1_{GMa\wedge c} \vee 1_{GNa\wedge c}$.

The proof for the lattice L satisfying (ii) is a repetition of the proof above, and hence we will omit it. For completing the proof of necessity we must show that $GH = GM \vee GN$. Let $GK \geq GM, GN$, and so for each $x \in L$, $xGK1_{GMx}$ and $xGK1_{GNx}$. According to (DK3), $xGK(1_{GMx} \vee 1_{GNx})$, whence $GK \geq GH$, and thus GH = GM$\vee$GN.

2°: Let GH be the join of relations GM and GN on L, and $A_{GHx} = [x, 1_{GMx} \vee 1_{GNx}]$. Let us remove from the Hasse diagram of L all the points and the lines incident to those points, which are meet-reducible in L. Remove further the chain C_0 containing the zero element of L, if such a chain exists. If the diagram graph thus obtained is empty, L was the chain C_0, and the theorem holds. If not, let us consider the graph D obtained. If it is a tree, where the degree of point 1 only can be 3 or greater, then there is nothing to prove: the chains of this tree are the factors $L_1, \dots, L_m$ in (i), as the elements of a finite distributive lattice can be uniquely represented as meets of meet-irreducibles.

Assume that D is a tree and there is a point $a \neq 1$ with the degree at least 3. Then there are in D two points x and y which are meet-irreducible in L. Let us consider the sublattice of elements $\{x \wedge y, x, y, a, z\}$ of L, where $z \in D$, and $a < q < z$ holds for no $q \in L$ (e.g. $a \prec z$); such an element z exists in L as D is a tree and $a \neq 1$ (see Fig. 1(a)). We define



(a) (b) (c)

Figure 1

a generalized normality relation GM as follows: RGMs $\Longleftrightarrow$ $\Longleftrightarrow$ $r = s$ or $\exists q \in L$ such that $r = y \wedge q$ and $s \leq z \wedge q$; obviously GM is a generalized normality relation on L. We define another relation FN analogously: tGNu $\Longleftrightarrow$ $t = u$ or $\exists p \in L$ such that $t = p \wedge x$ and $u \leq z \wedge p$. One can easily see that $[x \wedge y, 1_{GMx\wedge y} \vee 1_{GNx\wedge y}] = [x \wedge y, a]$, but it holds for each $GK \geq GM, GN$ that xGKz and yGKz, whence $x \wedge y$GKZ, as well. But $z \notin [x \wedge y, a]$, which is a contradiction. So in the tree D only the point 1 can have degree 3 or greater.

Assume that D is unconnected graph. Let x be the point of D such that $x \neq 1$, but all the points $h_1, \dots h_{n_x}$ which are joined by a line to x in D are less than x in L. As the chain C_0 has been removed, there are in L also elements that are less than x. On the other hand, as $x \neq 1$, there is also a meet-reducible element a in L satisfying $x \prec a$, and let the shortest meet-representation of a in terms of meet-irreducibles contain an element $z \in L$. As the chain C_0 has been removed, there is in L an element y such that $y \vee x = a$, or there are two non-comparable elements $u, y \leq x$ such that $x = u \vee y$ (see Figures 1(b) and 1(c)).

In the case of Figure 1(b) we define two generalized normality relations GM and GN as in the case above. There are not two non-comparable elements $b \geq x$ and $c \geq y$ such that $b \vee c = z$ and $b \wedge c = x \wedge y$, as in the other case $b \wedge a = x$, because $b \wedge c = x \wedge y$, $a \succ x$, $a \geq y$ and $c \geq y$. Hence $z \notin [x \wedge y, 1_{GMx\wedge y} \vee 1_{GNx\wedge y}]$, and we get the desired contradiction.

In the case of Figure 1(c), the relations GM and GN can be defined as follows: rGMs $\Longleftrightarrow$ $r = s$ or $\exists p \in L$ such that $u \wedge p = r$ and $a \wedge p \geq s$, and $tGNv \Longleftrightarrow t = v$ or $\exists f \in L$ such that $f \wedge y = t$ and $f \wedge a \geq v$. The assumption in the case of Figure 1(c) says that there are not two non-comparable elements $b \geq u$ and $c \geq y$ such that $b \vee c = a$ and $b \wedge c = u \wedge y$, as in the other case $b \vee x = a$ or $c \vee x = a$. Hence $a \notin [u \wedge y, 1_{GMu\wedge y} \vee 1_{GNu\wedge y}]$. So D must be a connected tree, where only the point 1 can have the degree 3 or greater. This completes the proof.

The following lemma gives a join construction for generalized normality relations in the general case.

Lemma 2. Let GM and GN be two generalized normality relations on a finite lattice L. Then the family $\mathcal{A}(GH) = \{[a, 1_{GMa} \vee 1_{GNa} \vee U_a] \mid a \in L\}$, where $U_a = \bigvee_{Sa} \{(1_{GMx} \vee 1_{GNx} \vee U_x) \wedge (1_{GMy} \vee 1_{GNy} \vee U_y) \mid Sa$ is the set of all pairs $x, y \in L$ for which $x \wedge y = a\}$, generates a generalized normality relation GH on L and $GH = GM \vee GN$.

Proof. As $U_{a\wedge c}$ contains at least the term $(1_{GMa} \vee 1_{GNa} \vee U_a) \wedge (1_{GMc} \vee 1_{GNc} \vee U_c)$, then $b \wedge d \in [a \wedge c, 1_{GMa\wedge c} \vee 1_{GNa\wedge c} \vee U_{a\wedge c}]$ and (DK2) holds for aGHb and cGHd. The other conditions hold obviously.

Let GP be a generalized normality relation on L such that $GP \geq GM, GN$. Then $xGP1_{GMx}$ and $xGP1_{GNx}$ for each $x \in L$, and so $xGP(1_{GMx} \vee 1_{GNx})$, as well. According to the property (DK2) and to the finiteness of L, also $xGPU_x$. Hence $xGP(1_{GMx} \vee 1_{GNx} \vee U_x)$ for each $x \in L$, and thus $GP \geq GH$. Consequently, $GH = GM \vee GN$, and the lemma follows.

The following lemma gives a construction for the join of normality relations analogous to the results in Theorem 2.

Lemma 3. Let M and N be two normality relations on a finite distributive lattice L. The family $\mathcal{A}(H) = \{[a, 1_{Ma} \vee 1_{Na} \vee W_a] \mid a \in L\}$, where $W_a = \bigvee_{S_a} \{(1_{Mx_1} \vee 1_{Nx_1}) \wedge (1_{Mx_2} \vee 1_{Nx_2}) \wedge \ldots \wedge (1_{Mx_n} \vee 1_{Nx_n}) \mid S_a$ is the set of all sequences $x_1, \ldots, x_n$ for which $a = x_1 \vee x_2 \vee \ldots \vee x_n, n \geq 2\}$, generates a normality relation H on L and $H = N \vee M$, if $L = L_1 \times L_2 \times \ldots \times L_m$, where L_i is a chain for each value of $i = 1, \ldots, m$.

Proof. Let us consider first the condition (DK4). Let aHb and cHd; we must show that $a \vee c \vee (b \wedge d) \leq a \vee c \vee \{(1_{Ma} \vee 1_{Na} \vee W_a) \wedge (1_{Mc} \vee 1_{Nc} \vee W_c)\} \in [a \vee c, 1_{Ma \vee c} \vee 1_{Na \vee c} \vee W_{a \vee c}]$. By applying the distributivity we see that $(1_{Ma} \vee 1_{Na} \vee W_a) \wedge (1_{Mc} \vee 1_{Nc} \vee W_c) = \{(1_{Ma} \vee 1_{Na}) \wedge (1_{Mc} \vee 1_{Nc})\} \vee \{W_a \wedge (1_{Mc} \vee 1_{Nc})\} \vee \{W_c \wedge (1_{Ma} \vee 1_{Na})\} \vee \{W_a \wedge W_c\} \leq W_{a \vee c}$ according to the definition of $W_{a \vee c}$. As $a \vee c \leq 1_{Ma \vee c} \vee 1_{Na \vee c}$, the assertion follows by combining these two observations.

(DK0), (DK1) and (DK3) hold obviously, and so we shall consider the condition (DK2) only. Let aHb and cHd. The relation H satisfies (DK2), if $b \wedge d \leq (1_{Ma} \vee 1_{Na} \vee W_a) \wedge (1_{Mc} \vee 1_{Nc} \vee W_c) \in [a \wedge c, 1_{Ma \wedge c} \vee 1_{Na \wedge c} \vee W_{a \wedge c}]$. As above, we consider the term $\{(1_{Ma} \vee 1_{Na}) \wedge (1_{Mc} \vee 1_{Nc})\} \vee \{W_a \wedge (1_{Mc} \vee 1_{Nc})\} \vee \{W_c \wedge (1_{Ma} \vee 1_{Na})\} \vee \{W_a \wedge W_c\} = (1_{Ma} \vee 1_{Na} \vee W_a) \wedge (1_{Mc} \vee 1_{Nc} \vee W_c)$. Similarly as in the proof of Theorem 1, we can show that

(1) $\quad (1_{Ma} \vee 1_{Na}) \wedge (1_{Mc} \vee 1_{Nc}) \leq 1_{Ma \wedge c} \vee 1_{Na \wedge c}$.

As $a \wedge c = (x_1 \wedge c) \vee (x_2 \wedge c) \vee \ldots \vee (x_n \wedge c)$ for each sequence

$x_1, x_2, \ldots, x_n$ with the property $x_1 \vee \ldots \vee x_n = a$, $W_{a\wedge c} \geq$ $\geq (1_{Mx_1\wedge c} \vee 1_{Nx_1\wedge c}) \wedge \ldots \wedge (1_{Mx_n\wedge c} \vee 1_{Nx_n\wedge c}) \geq \{(1_{Mx_1} \vee 1_{Nx_1}) \wedge$ $\wedge (1_{Mc} \vee 1_{Nc})\} \wedge \{(1_{Mx_2} \vee 1_{Nx_2}) \wedge (1_{Mc} \vee 1_{Nc})\} \wedge \ldots \wedge \{(1_{Nx_n} \vee$ $\vee 1_{Mx_n}) \wedge (1_{Mc} \vee 1_{Nc})\} = \{(1_{Mx_1} \vee 1_{Nx_1}) \wedge \ldots \wedge (1_{Mx_n} \vee 1_{Nx_n})\} \wedge$ $\wedge (1_{Mc} \vee 1_{Nc})$. By forming the join of all terms $\{(1_{Mx_1} \vee 1_{Nx_1}) \wedge$ $\wedge \ldots \wedge (1_{Mx_n} \vee 1_{Nx_n})\} \wedge (1_{Mc} \vee 1_{Nc})$, where $x_1 \vee \ldots \vee x_n = a$, we obtain the term $W_a \wedge (1_{Mc} \vee 1_{Nc})$, and as each member of the join was less or equal to $W_{a\wedge c}$, then

(2) $$W_{a\wedge c} \geq W_a \wedge (1_{Mc} \vee 1_{Nc}).$$

Similarly we see that

(3) $$W_{a\wedge c} \geq W_c \wedge (1_{Ma} \vee 1_{Na}).$$

Consider finally the term $W_a \wedge W_c$. Let $a = x_1 \vee \ldots \vee x_n$ and $c = y_1 \vee \ldots \vee y_m$, then $a \wedge c = (x_1 \wedge y_1) \vee (x_2 \wedge y_1) \vee \ldots$ $\vee (x_n \wedge y_1) \vee (x_1 \wedge y_2) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_2) \vee (x_1 \wedge y_3) \vee$ $\vee \ldots \vee (x_n \wedge y_m)$. According to the definition of $W_{a\wedge c} \geq$ $\geq (1_{Mx_1\wedge y_1} \vee 1_{Nx_1\wedge y_1}) \wedge (1_{Mx_2\wedge y_1} \vee 1_{Nx_2\wedge y_1}) \wedge \ldots \wedge (1_{Mx_n\wedge y_m} \vee$ $\vee 1_{Nx_n\wedge y_m})$. On the other hand,

$$(1_{Mx_1\wedge y_1} \vee 1_{Nx_1\wedge y_1}) \geq (1_{Mx_1} \vee 1_{Nx_1}) \wedge (1_{My_1} \vee 1_{Ny_1}),$$

$$(1_{Mx_2\wedge y_1} \vee 1_{Nx_2\wedge y_1}) \geq (1_{Mx_2} \vee 1_{Nx_2}) \wedge (1_{My_1} \vee 1_{Ny_1}),$$

$$\vdots$$

$$(1_{Mx_n\wedge y_1} \vee 1_{Nx_n\wedge y_1}) \geq (1_{Mx_n} \vee 1_{Nx_n}) \wedge (1_{My_1} \vee 1_{Ny_1}),$$

$$\vdots$$

$$(1_{Mx_n\wedge y_m} \vee 1_{Nx_n\wedge y_m}) \geq (1_{Mx_n} \vee 1_{Nx_n}) \wedge (1_{My_m} \vee 1_{Ny_m}),$$

and by forming the meets of both sides and by ordering the terms in the right side, we see that $W_{a\wedge c} \geq (1_{Mx_1\wedge y_1} \vee \vee 1_{Nx_1\wedge y_1}) \wedge \ldots \wedge (1_{Mx_n\wedge y_m} \vee 1_{Nx_n\wedge y_m}) \geq (1_{Mx_1} \vee 1_{Nx_1}) \wedge \wedge (1_{Mx_2} \vee 1_{Nx_2}) \wedge \ldots \wedge (1_{Mx_n} \vee 1_{Nx_n}) \wedge (1_{My_1} \vee 1_{Ny_1}) \wedge (1_{My_2} \vee \vee 1_{Ny_2}) \wedge \ldots \wedge (1_{My_m} \vee 1_{Ny_m})$.

By forming the join over all pairs $(x_1, \ldots, x_n)$ and $(y_1, \ldots \ldots, y_m)$, where $x_1 \vee \ldots \vee x_n = a$ and $y_1 \vee \ldots \vee y_m = c$, we see that

(4) $$W_{a\wedge c} \geq W_a \wedge W_c.$$

By combining now the results (1),(2),(3) and (4) obtained above, we see that $(1_{Ma} \vee 1_{Na} \vee W_a) \wedge (1_{Mc} \vee 1_{Nc} \vee W_c) \leq (1_{Ma\wedge c} \vee \vee 1_{Na\wedge c} \vee W_{a\wedge c})$. Obviously $a \wedge c \leq (1_{Ma} \vee 1_{Na} \vee W_a) \wedge (1_{Mc} \vee 1_{Nc} \vee \vee W_c)$, and the assertion follows. So H satisfies also (DK2), and hence H is a normality relation on L.

Let K be a normality relation on L such that $K \geq N, M$. According to (DK3), $xK(1_{Nx} \vee 1_{Mx})$ for each $x \in L$, and according to (DK4) and (DK3), $xK(x \vee W_x)$ for each $x \in L$. By applying (DK3) once again, we see that $xK(1_{Nx} \vee 1_{Mx} \vee W_x)$ for each $x \in L$, and hence $K \geq H$. Thus $H = N \vee M$, and the lemma follows.

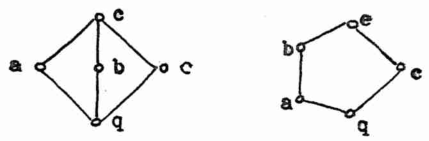
Now we can prove a theorem on the distributivity of the lattice GN(L).

<u>Theorem 3</u>. The lattice GN(L) of all generalized normality relations on a finite lattice is distributive if and only if L is distributive and $GH = GN \vee GM$ is determined by the family $\mathcal{A}(GH) = \{[x, 1_{GNx} \vee 1_{GMx}] \mid x \in L\}$.

<u>Proof</u>. Let L be a finite distributive lattice satisfying the condition of the theorem, and GK, GN and GM three generalized normality relations on L. It is sufficient to show that $GK \wedge (GN \vee GM) \leq (GK \wedge GN) \vee (GK \wedge GM)$, from which the

distributivity of GN(L) follows. Let $a \{GK \wedge (GN \vee GM)\} b \Longleftrightarrow$ $\Longleftrightarrow aGKb$ and $a(GN \vee GM)b$. Furthermore, $a(GN \vee GM)b \Longrightarrow b \in$ $\in [a, 1_{GNa} \vee 1_{GMa}]$, and so $b = b \wedge (1_{GNa} \vee 1_{GMa}) = (b \wedge 1_{GNa}) \vee$ $\vee (b \wedge 1_{GMa})$. Trivially, $a(GK \wedge GN)(b \vee 1_{GNa})$ and $a(GK \wedge GM)(b \vee$ $\vee 1_{GMa})$, which imply according to (DK3) that $a \{(GK \wedge GM) \vee$ $\vee (GK \wedge GN)\} b$. Thus $GK \wedge (GN \vee GM) = (GK \wedge GN) \vee (GK \wedge GM)$.

In the converse part we shall first show that L is necessarily distributive. If L is non-distributive, it contains as a sublattice at least one of the lattices L′ and L″ of Figure 2. Consider first the case of sublattice L′.



L′ L″

As L is finite, we can construct five normality relations such that the only nontrivial interval in the family $\mathcal{A}$ generating the relations is $[0,q]$, $[0,a]$, $[0,b]$, $[0,c]$ or $[0,e]$; we denote the corresponding relations by $G[0,q]$, $G[0,a]$, $G[0,b]$, $G[0,c]$ and $G[0,e]$. Clearly these relations form a non-distributive sublattice of the lattice GN(L) as $U_0 \leq q$. Similarly we see that the lattice GN(L) of a lattice L containing L″ as sublattice, contains a non-distributive sublattice. Hence L is distributive.

If the join $GH = GN \vee GM$ cannot be generated by the family $\mathcal{A}(GH) = \{[x, 1_{GMx} \vee 1_{GNx}] \mid x \in L\}$, we obtain the cases of the proof of Theorem 2 given in Figure 1. In the cases of Figure 1(a) and 1(b), we define GK as follows: $sGKu \Longleftrightarrow$ $\Longleftrightarrow s = u$ or $\exists t \in L$ such that $t \wedge (x \wedge y) = s$ and $t \wedge z \geq u$.

As L is distributive, GK is a generalized normality relation on L; GN and GM are defined similarly as in the proof of Theorem 2. So $(x\wedge y)\{GK\wedge(GN\vee GM)\}z$. According to the definition of GK, for each $d>x\wedge y$, $A_{KGd}=[d,d]$, and hence $U_{x\wedge y}=x\wedge y$ for $(GK\wedge GM)\vee(GK\wedge GN)$. On the other hand, the proof of Theorem 2 shows that there are not in L two non-comparable elements $b\geq x$ and $c\geq y$ such that $b\vee c=z$ and $b\wedge c=x\wedge y$, whence the relation $(x\wedge y)\{(GK\wedge GM)\vee$ $\vee(GK\wedge GN)\}z$ does not hold. The proof is similar in the case of Figure 1(c). This completes the proof.

3. On direct decompositions. At first we prove a theorem on direct decompositions by means of generalized normality relations.

Theorem 4. Let L be a finite lattice such that $L=$ $=L_1'\times L_2'\times\ldots\times L_m'$, where L_i' is a chain. L has a direct decomposition if and only if there are two nontrivial generalized normality relations GM, $GK\in GN(L)$ such that $GM\wedge GK=0$ and $GM\vee GK=1$ in GN(L).

Proof. 1^o: Let $L=L_1\times L_2$. We define two relations as follows: $aGMb\Longleftrightarrow a=(x_1,x_2)$, $b=(x_1,y_2)$ and $x_2\leq y_2$; $cGKd\Longleftrightarrow c=(z_1,z_2)$, $d=(w_1,z_2)$ and $z_1\leq w_1$. It is an exercise to show that GM and GK are generalized normality relations on L; we shall only show that GM and GK are complements in GN(L). Let $t\leq u$ in L, where $u=(u_1,u_2)$ and $t=$ $=(t_1,t_2)$. Then $(u_1,u_2)GM(u_1,t_2)$ and $(u_1,u_2)GK(t_1,u_2)$. Furthermore, $(t_1,u_2)\vee(u_1,t_2)=(u_1\vee t_1,u_2\vee t_2)=(t_1,t_2)$, and so the relations above imply $a(GK\vee GM)t$. Hence $GM\vee GK=1$. If $h(GM\wedge GK)f$, then according to the definition of GM,

$h_1 = f_1$ in $h = (h_1,h_2)$ and $f = (f_1,f_2)$. Similarly GK implies that $h_2 = f_2$, whence $(h_1,h_2) = (f_1,f_2) = h = f$. Thus $GK \wedge GM = 0$.

2^o: Let $GM \wedge GK = 0$ and $GM \vee GK = 1$ in GN(L). We shall show that $L = [0,1_{GKO}] \times [0,1_{GMO}]$. Each join-irreducible element of L belongs to one of the sets $[0,1_{GKO}]$, $[0,1_{GMO}]$. Indeed, assume that x is join-irreducible and $x \notin [0,1_{GKO}]$, $[0,1_{GMO}]$. Then $x \in [0,1_{GKO} \vee 1_{GMO}]$, as $GM \vee GK = 1$. So $x \wedge (1_{GKO} \vee 1_{GMO}) = (x \wedge 1_{GKO}) \vee (x \wedge 1_{GMO})$, from which it follows that x is join-reducible, or $1_{GKO} = 0$, or $1_{GMO} = 0$, and $x \in [0,1_{GMO}]$, or $x \in [0,1_{GKO}]$, respectively; a contradiction in each case. Furthermore, $GM \wedge GK = 0$, and so $[0,1_{GMO}] \cap [0,1_{GKO}] = \{0\}$. As L is finite and distributive, for each $z \in L$, z is the join of suitable join-irreducibles, i.e. $z = (\bigvee_i (q^z_{GK})_i) \vee (\bigvee_j (p^z_{GM})_j)$, where $(q^z_{GK})_i$ is a join-irreducible of $[0,1_{GKO}]$ and $(p^z_{GM})_j$ a join-irreducible of $[0,1_{GMO}]$. Clearly $\bigvee_i (q^z_{GK})_i = q^z_{GK} \in [0,1_{GKO}]$ and $\bigvee_j (p^z_{GM})_j = p^z_{GM} \in [0,1_{GMO}]$. We map z onto (q^z_{GK},p^z_{GM}). According to the uniqueness of the joinrepresentation by means of join-irreducibles in a distributive lattice, the mapping is a lattice morphism. If z has the figures (q^z_{GK},p^z_{GM}) and $(q^{z1}_{GK},p^{z1}_{GM})$, then the uniqueness of the joinrepresentation implies that $p^z_{GM} = p^{z1}_{GM}$ and $q^z_{GK} = q^{z1}_{GK}$. Similarly we see that each element of $[0,1_{GKO}] \times [0,1_{GMO}]$ has an image in L, and hence $L = [0,1_{GKO}] \times [0,1_{GMO}]$. This completes the proof.

As in the case of the preceding theorem GN(L) is distributive,one can prove the following generalization by an analogous way.

Corollary. Let L be a finite lattice, $L = L'_1 \times \ldots \times L'_m$, where $L'_1,\ldots,L'_m$ are chains. L has a direct decomposition with n factors if and only if there are n nontrivial generalized normality relations $GM_1, GM_2, \ldots, GM_n$ such that $GM_k \wedge GM_j = 0$ for each pair k,j, $k \neq j$, and $GM_1 \vee GM_2 \vee \ldots \vee GM_n = 1$ in $GN(L)$.

The following theorem gives the corresponding result in the case of normality relations.

Theorem 5. Let L be a finite lattice such that $L = L'_1 \times \ldots \times L'_m$, where $L'_1,\ldots,L'_m$ are chains. L has a direct decomposition if and only if there are two nontrivial normality relations $K, M \in N(L)$ such that $K \wedge M = 0$ and $K \vee M = 1$ in $N(L)$.

Proof. 1°: Let $L = L_1 \times L_2$. We define K and M similarly as the generalized normality relations of Theorem 4: $aKb \Longleftrightarrow a = (a_1,a_2)$, $b = (a_1,b_2)$ and $a_2 \leq b_2$; $cMd \Longleftrightarrow c = (c_1,c_2)$, $d = (d_1,d_2)$ and $c_1 \leq d_1$. We shall show that (DK4) holds for K; the proof is similar for M. Let aKb and fKh. Then $a \vee f = (a_1 \vee f_1, a_2 \vee f_2)$ and $h \wedge b = (a_1 \wedge f_1, b_2 \wedge h_2)$. Further, $a \vee f \vee (h \wedge b) = (a_1 \vee f_1 \vee (a_1 \wedge f_1), a_2 \vee f_2 \vee (b_2 \wedge h_2)) = (a_1 \vee f_1, a_2 \vee f_2 \vee (b_2 \wedge h_2))$. The first components of $a \vee f$ and $a \vee f \vee (h \wedge b)$ are the same and $a_2 \vee f_2 \leq a_2 \vee f_2 \vee (b_2 \wedge h_2)$, whence $(a \vee f)K(a \vee f \vee (h \wedge b))$. The other conditions hold obviously, and hence K and M are normality relations. The latter part of 1° is a repetition of 1° in the proof of Theorem 4, and hence we omit it.

2°: We shall show that the construction of the proof

2° of Theorem 4 holds. We must only show that each join-irreducible element x of L belongs to $[0, 1_{KO}]$ or to $[0, 1_{MO}]$; in fact, we show that $1_{KO} \vee 1_{MO} = 1$ in L. Let us consider the normality relation $K \vee M$. $A_{K \vee MO} = [0, 1_{KO} \vee 1_{MO} \vee W_{O}]$, and as the only join-expression for 0 is $0 = 0 \vee 0$, $W_{O} = (1_{KO} \vee 1_{MO}) \wedge (1_{KO} \vee 1_{MO})$, we see that $A_{K \vee MO} = [0, 1_{KO} \vee 1_{MO}]$. Furthermore, as $K \vee M = 1$ in $N(L)$, then $A_{K \vee MO} = L$, and hence $1_{KO} \vee 1_{MO} = 1$ in L. The rest is a repetition of the proof 2° in Theorem 4.

As we have not shown the distributivity of $N(L)$, the corollary of Theorem 4 need not hold in the case of normality relations.

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