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COGENERATION AND MINIMAL REALIZATION

(Preliminary communication)

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Abstract: Given a triple algebra (Q,d) and a quotient e of Q, then e is said to cogenerate the biggest quotient-algebra of (Q,d), contained in e, provided that such exists. (This is dual to the generation of subalgebras.) A necessary and sufficient condition on a triple **T** is exhibited in order that **T** admit cogeneration, i.e. that each quotient object on each **T**-algebra cogenerate something. The condition is very simple; the functor T must preserve cointersections. For triples over sets this characterizes finitary algebras.

Cogeneration is closely related to minimal realizations for triple machines. In terms of Arbib and Manes, an input process X is proved to admit minimal realization iff X@ preserves cointersections.

All the details are going to appear in 3 .

Key words: Triple algebra, generation of subalgebras, cogeneration of quotient algebras, preservation of cointersections, triple machines, minimal realization.

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A) Cogeneration

A,1 We assume that a category $\mathcal{K}$ is given, equipped with a factorization system ($\mathcal{E}$,$\mathcal{M}$). This allows us to speak about quotient objects of an object Q, as morphisms e: $Q \to \overline{Q}$ in $\mathcal{E}$ "up to isomorphism". The quotients of Q are naturally ordered: $e_1 \leq e_2$ iff $e_2 = k.e_1$ for some k.

The least upper bounds are called cointersections; if they always exist (even for classes of quotients), $\mathcal{E}$ is said to be closed to cointersections. And a functor, respecting and respecting these least upper bounds, is said to to preserve cointersections. (All this is dual to the usual notion of big intersections of subobjects.)

We consider a (fixed) triple $\mathbb{T} = (T, \mu, \eta)$, which will be supposed to preserve $\mathcal{E}$ (i.e., $e \in \mathcal{E}$ implies $Te \in \mathcal{E}$). A quotient algebra of a $\mathbb{T}$-algebra (Q,d) is a $\mathbb{T}$-homomorphism $h: (Q,d) \longrightarrow (Q',d')$ with $h \in \mathcal{E}$.

A,2 <u>Definition</u>: A triple $\mathbb{T}$ is said to <u>admit cogeneration</u> if for every $\mathbb{T}$-algebra (Q,d) and every quotient object e of Q there exists the biggest quotient algebra c of (Q,d) with $c \leq e$. Then c is said to be cogenerated by e.

<u>Note</u>. The cogeneration of quotient algebras is dual to the generation of subalgebras. If $\mathcal{K}$ has (big) intersections, the generation presents no problem: each subobject generates the intersection of all subalgebras, containing it. Fortunately, the intersection of $\mathbb{T}$-algebras is always a $\mathbb{T}$-algebra (for the forgetful functor $\mathcal{K}^{\mathbf{T}} \longrightarrow \mathcal{K}$ creates limits). Now, assume that T preserves cointersections. Then the cointersection of $\mathbb{T}$-algebras is always a $\mathbb{T}$-algebra (fpr the forgetful functor $\mathcal{K}^{\mathbf{T}} \longrightarrow \mathcal{K}$ creates all colimits, preserved by T). Thus, each quotient object cogenerates the cointersection of all quotient algebras, contained in it. This can be reversed as follows.

<u>Main Theorem</u>. Let $\mathcal{E}$ be closed to cointersections and let T preserve $\mathcal{E}$. Then $\mathbb{T}$ admits cogeneration iff T preserves cointersections.

A,3 <u>Corollary</u>. A triple over the category of sets admits cogeneration iff it is isomorphic to the W-free algebra triple for some variety W of finitary algebras.

Note. More on functors, preserving cointersections, can be found in [2]. E.g., under additional, rather mild, assumptions on $\mathcal{K}$, each functor which preserves cointersections, generates a free triple. (Recall from [5] that a free triple $\mathbb{T}$, generated by an endofunctor X, is a transformation t: X $\longrightarrow$ T such that for every triple $\mathbb{T}'$ and every transformation t′: X $\longrightarrow$ T′ there exists a unique triple morphism r: $\mathbb{T} \longrightarrow \mathbb{T}'$ with t′ = r.t.) A corollary: every triple which admits cogeneration, is a retract of a free triple.

Another result in [2] concers endofunctors of the category of vector spaces (over an arbitrary given field) from which we get

Corollary. A triple over vector spaces admits cogeneration iff it is finitary, i.e. T preserves filteres colimits.

B) Minimal realization

B1 Arbib and Manes investigate automata over free triples [4]. In the same direction, automata over arbitrary triples can be defined (cf. [4,6] and, for a more general approach,[7]). Concerning the minimal realization problem, this generalization of the Arbib-Manes approach turns out to be very convenient: the whole technique becomes much simpler.

Let X: $\mathcal{K} \longrightarrow \mathcal{K}$ be a functor, generating a free triple $\mathbf{T}$. (Arbib and Manes call X an input process and they denote $TQ = X^{@} Q$.) Then pairs (Q,δ), where Q is an object and $\delta : XQ \longrightarrow Q$ is a morphism, naturally correspond to $\mathbf{T}$-algebras (Q,d); therefore, $\mathbf{T}$-algebras will play the role of (Q,δ) for triple machines.

B2 As in A) above, we have $\mathcal{K}$, $(\mathcal{E},\mathcal{M})$ and $\mathbf{T}$. For fixed objects Y and I, a machine is tuple $M = (Q,d,Y,\beta,I,\tau)$, where (Q,d) is a $\mathbf{T}$-algebra and $\beta : Q \longrightarrow Y$ and $\tau : I \longrightarrow Q$ are morphisms. $\mathbf{T}$-homomorphisms, commuting with both the β's and the τ's, are called simulations (from one machine to another).

Given a machine M, the morphism $r = d.T\tau : TI \longrightarrow Q$ is called the run map of M, and the composition $f_M = \beta.r: TI \longrightarrow Y$ is the behavior of M. The machine M is reachable if $r \in \mathcal{E}$.

A realization of a "behavior", i.e. of a morphism $f: TI \longrightarrow Y$, is any machine M with $f_M = f$. This realization is minimal if (i) it is reachable and (ii) for every reachable realization M' there exists a unique simulation from M' to M. Every behavior has a reachable realization, e.g. $M(f) = (TI, \mu^I, Y, f, I, \eta^I)$ - here $r = id_{TI}$. The problem of minimal realization is: does every behavior have a minimal realization? If this is so (for all I, Y and f) then $\mathbf{T}$ is said to admit minimal realization.

B,3 Theorem. A triple $\mathbf{T}$, preserving $\mathcal{E}$, admits cogeneration iff it admits minimal realization.

Combining this theorem with the above result, we obtain, in the terminology of Arbib abd Manes:

Corollary. Let $\mathcal{E}$ be closed to cointersection and let X be an input process, preserving $\mathcal{E}$. Then X admits minimal realization iff $X^{@}$ preserves cointersections.

B,4 A finite model. To capture also finite-state machines, we can proceed as in [2], starting with a class $\mathcal{E}$ of epis. We do not assume any factorization properties and we think of $\mathcal{E}$ -morphisms as "finite quotients". A behavior is regular if it has a reachable realization (i.e., $r \in \mathcal{E}$ - recall that if $\mathcal{E}$ contains all isomorphisms, then all behaviors f are regular, via M(f)). And $\mathbf{T}$ admits minimal realization if each regular (!) behavior has a minimal realization. As above, it suffices that $\mathcal{E}$ is closed to, and T preserves, cointersections. This can be reversed if $T(\mathcal{E}) \subset \mathcal{E}$:

Theorem. Let $\mathcal{E}$ be a class of epis, closed to cointersections, let T preserve $\mathcal{E}$. Then $\mathbf{T}$ admits minimal realization iff T preserves cointersections.

Example. Let $\mathcal{K}$ be the category of sets, or, the category of vector spaces. Let $\mathcal{E}$ denote epis $e: A \longrightarrow \bar{A}$ with $\bar{A}$ finite (resp., finite-dimensional). Then $\mathcal{E}$-cointersections are proved in [2] to be absolute colimits.

Corollary. Every triple over sets or over vector spaces admits finite minimal realization.

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ST 18