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A NOTE ON NORMAL TOPOLOGICAL FUNCTORS AND EXTENSIONS OF TRANSFORMATIONS

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Abstract: The notion of normality can be variously generalized for the functors F: k $\longrightarrow$ Top from a category k into topological spaces (by means of the separation of the closed subfunctors of F by the open ones, the extensions of transformations of a closed subfunctor of F on the entire F, etc.). The discussion of the definitions is presented. The notion of the weakly filtered category is introduced and used (a category is weakly filtered if for any two morphisms $\alpha_1: \sigma \longrightarrow \sigma_1$, $\alpha_2: \sigma \longrightarrow \sigma_2$ there are morphisms $\beta_1: \sigma_1 \longrightarrow p$, $\beta_2: \sigma_2 \longrightarrow p$ with $\beta_1 \alpha_1 = \beta_2 \alpha_2$).

Key words: Topological functor, natural transformation.

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The notion of normality in topological spaces can be naturally transferred into the topological functors. There are some possibilities for the definition. We may call a functor F: k $\longrightarrow$ Top

SEP-normal if any two disjoint closed subfunctors of F can be separated by two open subfunctors,

TU-normal if any natural transformation of a closed subfunctor of F to the constant functor C_R on reals can be extended on the entire F,

TU_0^1-normal if any two disjoint closed subfunctors of F can be separated by a natural transformation,

OB-normal if $F\sigma$ is a normal topological space for all objects $\sigma \in k$.

This note brings a discussion of the relations between the present definitions (if a category k is given). Evidently, TU-normality is equivalent to TU_0^1-normality. The other notions are not the same. It is proved that TU-normality is equivalent to SEP-normality iff the category k has the following property: If $\alpha_1 : \sigma \longrightarrow \sigma_1$, $\alpha_2 :$ $: \sigma \longrightarrow \sigma_2$ are morphisms of k then there are morphisms $\beta_1 : \sigma_1 \longrightarrow p$, $\beta_2 : \sigma_2 \longrightarrow p$ such that $\beta_1 \alpha_1 = \beta_2 \alpha_2$. We call such categories weakly filtered (as a weaker notion than the one of the filtered category studied in Mac Lane's book [ML]).

For the small categories we have as a corrolary that 1. if k is weakly filtered and F is SEP-normal then colim F is a normal space and 2. the functor colim: : [k,Set] $\longrightarrow$ Set preserves monomorphisms iff k is weakly filtered.

The notion of OB-normality is completely different from the other ones. The situation is illustrated by examples.

I would like to thank V. Trnková who called my attention to this question. She also started the examination of similar problems (see [KT]). For further investigation of analogous sort, see [AR].

Notions and results. We shall deal with covariant functors $F: k \longrightarrow Top$ from a category k into the category of topological spaces. A functor $F_1: k \longrightarrow Top$ is a subfunctor (closed subfunctor, open subfunctor) of F if for each morphism $\alpha: \sigma \longrightarrow p$, $F_1\sigma$ is a subspace (closed subspace, open subspace) of $F\sigma$ and $F_1\alpha$ is the domain-range restriction of $F\alpha$. A subfunctor F_1 is inversion preserving (or IP-subfunctor) if $x \in F_1\sigma$ whenever $F\alpha(x) \in F_1\sigma_1$ for a morphism $\alpha: \sigma \longrightarrow \sigma_1$.

Two subfunctors F_1, F_2 of F are separated if there exist two disjoint open subfunctors G_1, G_2 such that $F_1 \subset G_1$, $F_2 \subset G_2$. If any two disjoint closed subfunctors of F are separated we call F SEP-normal.

Following the topological situation, we can call a functor $F: k \longrightarrow Top$ Tietze-Urysohn normal (TU-normal) if it holds: Given a closed subfunctor F_1 of F and given a natural transformation $\tau_1: F_1 \longrightarrow C_R$, C_R being the constant functor on reals, then there exists a transformation $\tau: F \longrightarrow C_R$ which is an extension of τ_1. Of course, any TU-normal functor is SEP-normal.

<u>Definition</u>: A category is called weakly filtered if for each pair of morphisms $\alpha_1: \sigma \longrightarrow \sigma_1$, $\alpha_2: \sigma \longrightarrow \sigma_2$ there are morphisms $\beta_1: \sigma_1 \longrightarrow p$, $\beta_2: \sigma_2 \longrightarrow p$ with $\beta_1\alpha_1 = \beta_2\alpha_2$.

<u>Theorem</u>: Let k be a category. If k is weakly filtered then any SEP-normal functor $F: k \longrightarrow Top$ is TU-normal. If k is not weakly filtered then we can construct a SEP-normal functor $F: k \longrightarrow Top$ which is not TU-normal.

Proof: Suppose k is weakly filtered and $F: k \longrightarrow Top$

is SEP-normal. Then any two disjoint closed subfunctors of F may be separated by two open (or closed) IP-subfunctors of F. Indeed, if F_1, F_2 are two closed subfunctors of F then they are separated by open subfunctors H_1, H_2 and putting

$$G_1\sigma = \{x \in F\sigma \mid F\alpha(x) \in H_1 p \text{ for some } \alpha : \sigma \longrightarrow p\}$$

$$G_2\sigma = \{y \in F\sigma \mid F\alpha(y) \in H_2 p \text{ for some } \alpha : \sigma \longrightarrow p\}$$

we obtain two open IP-subfunctors with $G_1 \supset F_1$, $G_2 \supset F_2$. We have to show that G_1, G_2 are disjoint. Suppose $z \in G_1\sigma \cap \cap G_2\sigma$. Then there are morphisms $\alpha_1 : \sigma \longrightarrow \sigma_1$, $\alpha_2 : : \sigma \longrightarrow \sigma_2$ with $F\alpha_1(z) \in H_1\sigma_1$, $F\alpha_2(z) \in H_2\sigma_2$. Choose $\beta_1 : \sigma_1 \longrightarrow p$, $\beta_2 : \sigma_2 \longrightarrow p$ such that $\beta_1\alpha_1 = \beta_2\alpha_2$. Then $F\beta_1 F\alpha_1(z) = F\beta_2 F\alpha_2(z)$ and therefore $H_1 p \cap H_2 p \neq \emptyset$ - a contradiction.

If we want to separate F_1, F_2 by closed IP-subfunctors we first take the open IP-subfunctors G_1, G_2 and put $K_1\sigma = F\sigma - G_1\sigma$. Then we separate K_1, F_1 by open IP-subfunctors G_1', G_2' and put $K_2\sigma = F\sigma - G_1'\sigma$. The functors K_1, K_2 will do.

According to the Urysohn's procedure, it suffices to prove that for any disjoint closed subfunctors F_1, F_2 of F and for any transformation $\tau : F_1 \cup F_2 \longrightarrow R$ with $\tau' F_1 = 0$, $\tau' F_2 = 1$ there is an extension on F. But we can adopt the standard method - the role of the open sets in the sequence from one closed set to the other play the open IP-subfunctors of F (the induction runs by the observations on the start of this proof).

Conversely, suppose k is not weakly filtered. So there exist morphisms $\alpha_1 : \sigma \longrightarrow \sigma_1$, $\alpha_2 : \sigma \longrightarrow \sigma_2$ such that $\beta_1 \alpha_1 \neq \beta_2 \alpha_2$ for all morphisms β_1, β_2. Let $F : k \longrightarrow Set$ be the functor Hom $(\sigma, -)$, i.e. $F\,p = \{\alpha \mid \alpha : \sigma \longrightarrow p\}$. Endow each F p with the discrete topology and define F_1, F_2 such that

$$F_1 p = \{\alpha : \sigma \longrightarrow p \mid \alpha = \beta \alpha_1 \text{ for some } \beta : \sigma_1 \longrightarrow p\}$$

$$F_2 p = \{\alpha : \sigma \longrightarrow p \mid \alpha = \beta \alpha_2 \text{ for some } \beta : \sigma_2 \longrightarrow p\}.$$

The functors F_1, F_2 are disjoint (closed, open) subfunctors of F. Consider the transformation $\tau' : F_1 \cup F_2 \longrightarrow C_R$ such that $\tau' F_1 = 0$, $\tau' F_2 = 1$. If $\tau : F \longrightarrow C_R$ is an extension of τ' then $0 = \tau'_{\sigma_1}(\alpha_1) = \tau_\sigma(\mathrm{id}_\sigma) = \tau'_{\sigma_2}(\alpha_2) = 1$ - a contradiction.

Remark. A monoid (as a category) is weakly filtered iff the intersection of each pair of its left ideals is non-void.

A partially ordered set (as a thin category) is weakly filtered iff every its component is directed.

Proof is easy.

One more definition of normality may be in place: A functor $F : k \longrightarrow Top$ is called OB-normal if $F\sigma$ is a normal space for all objects $\sigma \in k$. As the following examples show, the situation here is less nice than that in the Theorem before.

Statement 1: Let k be a finite category. If k is weakly filtered then any OB-normal functor $F : k \longrightarrow Top$ is TU-normal.

Proof is not difficult.

Statement 2: Let k be a partially ordered set. If every component of k has the greatest element then any OB-normal functor $F: k \longrightarrow Top$ is TU-normal. If a component of k has not the greatest element then we can construct an OB-normal functor $F: k \longrightarrow Top$ which is neither SEP-normal nor TU-normal.

Proof: Evidently, the first part holds. The second part will be proved in two steps. First, let ω be a limit ordinal and let Ord_ω be the set of all smaller ordinals than ω. Define an OB-normal functor $F: Ord_\omega \longrightarrow Top$ as follows: Given a morphism $\alpha: \sigma \longrightarrow p$ (i.e. $\sigma \leq p$) then $F\sigma = Fp = Ord_\omega \vee \{\omega\}$. The topology is discrete on the subspace Ord_ω and a base of the neighbourhoods of ω is formed by the sets $\mathcal{O}_q = \{\sigma \in Ord_\omega \mid \sigma > q\} \cup \{\omega\}$. The mapping $F\alpha : F\sigma \longrightarrow Fp$ is defined such that if $q < \sigma$ or $q > p$ then $F\alpha(q) = q$, $F\alpha(q) = 0$ otherwise. Further define subfunctors F_1, F_2 such that $F_1\sigma = \{0\}$, $F_2\sigma = \{\omega\}$ for any $\sigma \in k$. Finally, define a transformation $\tau': F_1 \cup F_2 \longrightarrow C_R$ such that $\tau' F_1 = 0$, $\tau' F_2 = 1$. It is easy to check that τ' has no extension on F.

Let $k = (X, \leq)$. We can assume that $(X, \leq)$ is connected and directed. Take a maximal chain $(X', \leq)$ in $(X, \leq)$ with respect to the ordering $\leq$. The chain has a cofinal subset $(Y, \leq)$ equivalent to the set Ord_ω for a limit ordinal ω. By the previous observation, we have an OB-normal functor $F: (Y, \leq) \longrightarrow Top$, a closed subfunctor F' of F and a transformation $\tau': F' \longrightarrow C_R$ which cannot be extended on F. We extend F on the entire $(X, \leq)$. Let $\alpha: \sigma \longrightarrow p$ (i.e. $\sigma \leq p$) and $\sigma, p \in X$. Then put $H\alpha = F\beta$ where $\beta: \sigma' \longrightarrow p'$, $\sigma', p' \in Y$ and σ' (and similarly p') is determined by the following condition: σ' is the smallest element of $(Y, \leq)$ among those which are not smaller than σ. It is easy to check that H is a functor and the proof is finished in fact.

<u>Statement 3</u>: Let k be a group. If k has a regular cardinality then there is an OB-normal functor $F: k \longrightarrow \mathrm{Top}$ which is not TU-normal. If k is finite then $F: k \longrightarrow \mathrm{Top}$ is OB-normal iff it is TU-normal.

Proof: Let G be an infinite group with regular cardinality. Take a well-ordering $\prec$ of G such that all sequents have a smaller cardinality than G. We shall define a functor $F: G \longrightarrow \mathrm{Top}$. Put $F(G) = H \times H - \{n, n\}$ where H is a space on the set $G \vee \{n\}$ such that the topology of H is discrete on G and a base of neighbourhoods of n is formed by the sets $\sigma_h = \{g \in G \mid g > h\} \cup \{n\}$. If $g \in G$ then we define F g such that $Fg(x,y) = (gx, y)$ if $x \neq n$, $Fg(n,y) = (n,y)$. Clearly F is an OB-normal functor (Fg is continuous as k has a regular cardinality). Define a closed subfunctor F' of F such that $F' = F_1 \cup F_2$ where $F_1 G = \{(n,g) \mid g \in G\}$ and $F_2 G = \{(g,n) \mid g \in G\}$. One can check that the transformation $\tau': F' \longrightarrow C_R$ such that $\tau' F_1 = 0$, $\tau' F_2 = 1$ has no extension on F.

R e f e r e n c e s :

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