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ON THE $f(H,K)p$ - theorems

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Abstract: The proof of a common generalization of the following theorems: An ovaloid with $Kp^2 = 1$ or $Hp = 1$ resp. is a sphere.

Key words: Ovaloid, support function, sphere, elliptic system.

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One of the theorems of R. Schneider [2] has two following corollaries: $M \subset E^3$ being an ovaloid with $Kp^2 = 1$ or $Hp = 1$ resp., it is a sphere; see [1], p. 61. We are going to prove another general theorem, the Kp- and Hp-theorems being its special cases.

Let $G \subset \mathbb{R}^2$ be a bounded domain, ∂G its boundary, $m: G \cup \partial G \longrightarrow E^3$ a surface, $S \in E^3$ a fixed point. For each $g \in G \cup \partial G$, define the vector $v(g)$ by

(1) $$S = m(g) + v(g) .$$

Let v_3 be a fixed field of unit normal vectors of the surface $m \equiv m(G \cup \partial G)$, the support function $p(g)$ be defined by

(2) $$p(g) = \langle v_3(g), v(g) \rangle .$$

Further, let

$$\sigma(g)^2 = |v(g)|^2 - p(g)^2 \quad ; \tag{3}$$

$\sigma(g) \geqq 0$ is, of course, the length of the orthogonal projection of $v(g)$ into the tangent plane of m at $m(g)$. Let us remark that the mean curvature H of m depends on the chosen field of unit normal vectors; nevertheless, Hp is an invariant.

__Theorem.__ __Let the situation be as above, and let__ F, M, $N : G \cup \partial G \longrightarrow \mathbb{R}$ __be functions. Suppose__: (i) $\sigma(g) = 0$ __for each__ $g \in \partial G$; (ii) __on__ $G \cup \partial G$,

$$Mp(Kp - H) + N(Hp - 1) = \sigma^2 F \, , \tag{4}$$

$$Kp^2M^2 + 2\, Hp\, MN + N^2 > 0 \, . \tag{5}$$

__Then__ m __is a part of a sphere with the center__ S.

__Proof.__ On m, consider a field of orthonormal frames $\{m, v_1, v_2, v_3\}$. Then

$$dm = \omega^1 v_1 + \omega^2 v_2 \, , \quad dv_1 = \omega_1^2 v_2 + \omega_1^3 v_3 \, , \tag{6}$$

$$dv_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 \, , \quad dv_3 = -\omega_1^3 v_1 - \omega_2^3 v_2$$

with the usual integrability conditions. From $\omega^3 = 0$,

$$\omega_1^3 = a\,\omega^1 + b\,\omega^2 \, , \quad \omega_2^3 = b\,\omega^1 + c\,\omega^2 \tag{7}$$

with

(8) $$2H = a + c , \quad K = ac - b^2 .$$

Write

(9) $$v = xv_1 + yv_2 + pv_3 ;$$

from (1) and $dS = 0$,

(10) $$dx - y\,\omega_1^2 - p\,\omega_1^3 + \omega^1 = 0 ,$$

$$dy + x\,\omega_1^2 - p\,\omega_2^3 + \omega^2 = 0 ,$$

$$dp + x\,\omega_1^3 + y\,\omega_2^3 = 0 .$$

On G , introduce isothermic coordinates (u,v) such that

(11) $I = r^2(du^2 + dv^2)$, $r(u,v) > 0$, i.e., $\omega^1 = r\,du$, $\omega^2 = r\,dv$.

Then

(12) $$\omega_1^2 = r^{-1}(-r_v du + r_u dv) .$$

From (10) and (7),

(13) $$x_u + r^{-1}r_v y = (pa - 1)r , \quad x_v - r^{-1}r_u y = pbr ,$$

$$y_u - r^{-1}r_v x = pbr , \quad y_v + r^{-1}r_u x = (pc - 1)r .$$

From $(13_{2,3})$,

(14) $$x_v - y_u = -r^{-1}r_v x + r^{-1}r_u y .$$

Multiplying $(13_{1,2,4})$ by $Mcp + N$, $-2\,Mbp$, $Map + N$ resp. and adding them together, we get

$$(15) \qquad (Mcp + N)x_u - 2\,Mbpx_v + (Map + N)y_v +$$

$$+ r^{-1}r_u(Map + N)x + \{r_v(Mcp + N) + 2\,r_u Mbp\}\,r^{-1}y =$$

$$= 2\,r\{Mp(Kp - H) + N(Hp - 1)\} \quad .$$

Now, $\sigma^2 = x^2 + y^2$, and the right-hand side of (15) may be written, because of (4), as $2\,r\,xF\,.\,x + 2\,ry\,F.y$. Thus (15) takes the form

$$(16) \qquad (Mcp + N)x_u - 2\,Mbpx_v + (Map + N)y_v = (.)x + (.)y \;.$$

Consider the system (14) + (16). It has the form

$$(17) \qquad a_{i1}x_u + a_{i2}x_v + b_{i1}y_u + b_{i2}y_v = c_{i1}x + c_{i2}y$$

$$(i = 1,2) \;.$$

Recall that (17) is called elliptic if the form

$$(18) \qquad \phi = (a_{12}b_{22} - a_{22}b_{12})\,\mu^2 - (a_{11}b_{22} - a_{21}b_{12} +$$

$$+ a_{12}b_{21} - a_{22}b_{11})\,\mu\nu \; + (a_{11}b_{21} - a_{21}b_{11})\,\nu^2$$

is definite; (17) being elliptic, $x = y = 0$ on ∂G induces $x = y = 0$ in G . In our case,

$$(19) \qquad \phi = (Map + N)\mu^2 + 2\,Mbp\,\mu\nu \; + (Mcp + N)\nu^2 \;;$$

the discriminant of (19) being exactly the left-hand side of (5), the system (14) + (16) is elliptic. From (i), $x = y = 0$ on ∂G . Thus $x = y = 0$ in G and $p =$ $=$ const. because of (10_3). QED.

<u>Corollary.</u> <u>Let the situation be as in the introduction, and let</u> $F\colon G \cup \partial G \longrightarrow E^3$ <u>be a function. Suppose:</u>

(i) $\sigma(g) = 0$ for each $g \in \partial G$; (ii) on $G \cup \partial G$,

(20) $$Hp - 1 = \sigma^2 F$$

or

(21) $$Kp^2 - 1 = \sigma^2 F , \quad Kp^2 + 2\, Hp + 1 > 0$$

resp. Then m is a part of a sphere.

Proof. In our Theorem, take $M = 0$, $N = 1$ or $M = N = 1$ resp. QED.

Thus we get natural generalizations of the Hp- and Kp-theorems resp.

R e f e r e n c e s

[1] O. GIERING: Globale Differentialgeometrie (Vorlesung im SS 1969).Univ. Stuttgart, Math.Inst. B.

[2] R. SCHNEIDER: Eine Kennzeichnung der Kugel. Arch. Math. 16, 235-240, 1965.

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