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RINGS ON CERTAIN CLASSES OF TORSION-FREE ABELIAN GROUPS

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Abstract: In earlier papers (R. Ree and R.J. Wisner, Proc. Amer. Math. Soc. 7(1956), 6-8 and B.J. Gardner, Comment. Math. Univ. Carolinae 15(1974), 381-392) the nil completely decomposable torsion-free abelian groups were characterized, and a description of the absolute annihilators of completely decomposable torsion-free abelian groups was given. For a completely decomposable torsion-free abelian group A, a chain

$0 \subseteq A(1) \subseteq A(2) \subseteq \ldots \subseteq A(\alpha) \subseteq \ldots \subseteq A(\mu) = A(\mu + 1)$

of "iterated absolute annihilators" of A was also defined, and this gave some information about the kinds of ring multiplications admitted by A. This paper is concerned with studying these same concepts for other classes of torsion-free abelian groups. § 2 is devoted to vector groups and certain direct products of slender groups, while § 3 deals with separable groups.

Key words: Ring, nil group, absolute annihilator.

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1. Preliminaries. Throughout this paper we use the word "group" to mean abelian group, and the word "ring" to mean a not necessarily associative ring. A ring $(\mathcal{R}, \times)$ with additive group isomorphic to A is called a ring on A. The annihilator of a ring $(\mathcal{R}, \times)$ is denoted by $(0:(\mathcal{R}, \times))$, and the absolute annihilator $A(*)$ of a group A is defined as the intersection of the annihilators of all rings $(\mathcal{R}, \times)$ on A.

Szele [8] defines the nil-degree (Nilstufe) of a group A as the largest integer n such that there is an associative ring $(\mathcal{R},\times)$ on A with $(\mathcal{R},\times)^n \neq 0$, if such an n exists. Analogously the first author [4] defined the <u>strong nil-degree</u> pf A as the largest integer n (if one exists) such that there is a ring $(\mathcal{R},\times)$ on A with $(\mathcal{R},\times)^n$, the subring generated by all products of the form $(\ldots((a_1\times a_2)\times a_3)\ldots\times a_n$, nonzero. We call a group A <u>nil</u> (resp. <u>strongly nil</u>) if A has nil-degree 1 (resp. strong nil-degree 1).

The type of an element a, or a rational group A is denoted by T(a), T(A) respectively. If A_1 and A_2 are two rational groups, then the product $T(A_1)\ T(A_2)$ and quotient $T(A_1):T(A_2)$ of the two types $T(A_1)$, $T(A_2)$ are defined as in [2]. All other unexplained notation appears in [1] or [2].

Ree and Wisner [6] have classified the nil completely decomposable torsion-free groups, a paraphrase of their result being:

If $A = \bigoplus_{i\in I} A_i$, where the A_i are rational groups, then A is nil (equivalently strongly nil) if and only if $T(A_i)\ T(A_j) \nleq T(A_k)$ for all i, j, and $k \in I$.

In the sequel we will need

<u>Proposition 1.1.</u> Let $A = \bigoplus_{i\in I} A_i$, where the A_i are rational groups. If $T(A_i)\ T(A_j) \leq T(A_k)$ for some i, j and $k \in I$ then there is an associative ring $(\mathcal{R},\times)$ on A with $A_i \times A_\ell \neq 0$ for some $\ell \in I$, and $A_m \times A_\ell = 0$ for all $m \in I$, $m \neq i$.

Proof: See the proof of Theorem 1.1 of [4].

2. <u>Vector groups</u>. A <u>vector group</u> is a direct product of rank one torsion-free groups (i.e., a group $V = \prod_{i\in I} R_i$ where the R_i are rational groups).

We begin this section by giving a description of the nil vector groups. To do this we need the following definitions, and the well known results (2.1) to (2.3).

A slender group A is a torsion-free group with the property that every homomorphism from a countable direct product of infinite cyclic groups $\langle e_n \rangle$ $(n = 1,2,\dots)$ into A sends almost all components $\langle e_n \rangle$ into the zero of A.

A set is measurable if I admits a countably additive measure μ such that μ assumes only the values 0 and 1, and

$$\mu(I) = 1, \quad \mu(i) = 0 \text{ for all } i \in I.$$

(2.1) (Sąsiada [7], Nunke [5]) Every countable and reduced torsion-free group is slender.

(2.2) (Fuchs [2], p. 160) Direct sums of slender groups are slender.

(2.3) (Łoś; see [2], pp. 161, 162) If G is a slender group, A_i $(i \in I)$ are torsion-free groups and the index set I is not measurable, then

(i) if ϕ is a homomorphism from $\prod_{i \in I} A_i$ into G such that $\phi(\bigoplus_{i \in I} A_i) = 0$, then $\phi = 0$;

(ii) there is a natural isomorphism

$$\mathrm{Hom}(\prod_{i \in I} A_i, G) \cong \bigoplus_{i \in I} \mathrm{Hom}(A_i, G).$$

Whenever we represent a vector group as a direct product $V = \prod_{i \in I} R_i$ in this section it is to be understood that the R_i are rational groups.

We are now in a position to prove

Lemma 2.4. If $V = \prod_{i \in I} R_i$ is a vector group such that the index set I is not measurable, every R_i is reduced and

$\mathrm{Hom}(R_i, \bigoplus_{j\in I} \mathrm{Hom}(R_j,R_k)) \neq 0$ for some i and $k\in I$, then there exists $j\in I$ with $T(R_i)\ T(R_j) \leqq T(R_k)$.

Proof: $\mathrm{Hom}(R_i, \bigoplus_{j\in I} \mathrm{Hom}(R_j,R_k))$ is a subgroup of $\mathrm{Hom}(R_i, \prod_{j\in I} \mathrm{Hom}(R_j,R_k))$ so $\mathrm{Hom}(R_i,\mathrm{Hom}(R_j,R_k)) \neq 0$ for some $j\in I$. Now $\mathrm{Hom}(R_j,R_k)$ is a rank one torsion-free group whose type is $T(R_k): T(R_j)$. Thus $T(R_i)\ T(R_j) \leqq [T(R_k): T(R_j)]\ T(R_j) \leqq T(R_k)$, as required.

Theorem 2.5. Let $V = \prod_{i\in I} R_i$ be a vector group where the index set I is not measurable. Then the following conditions are equivalent:

(1) V is strongly nil;

(2) V is nil;

(3) $T(R_i)\ T(R_j) \nleqq T(R_k)$ for all i, j and $k\in I$.

Proof: (1) $\Longrightarrow$ (2) is immediate.

(2) $\Longrightarrow$ (3). Suppose $T(R_i)\ T(R_j) \leqq T(R_k)$ for some i, j and $k\in I$. It follows from Proposition 1.1 that we can define a non-trivial associative ring on a completely decomposable direct summand V' of V. This ring can be extended to the whole of V by making all other products zero, so V is not nil.

(3) $\Longrightarrow$ (1). If V is not strongly nil, then $\mathrm{Hom}(V, \mathrm{Hom}(V,V)) \neq 0$.

Since $T(R_i)^2 \nleqq T(R_i)$ for all $i\in I$, and I is not measurable, (2.1) and (2.3)(ii) show that $\mathrm{Hom}(V,V) \cong \prod_{k\in I} \bigoplus_{j\in I} \mathrm{Hom}(R_j,R_k)$. Now $\mathrm{Hom}(R_j,R_k)$ is either zero or a rank one torsion-free group whose type is less than or equal to $T(R_k)$. (2.1) and (2.2) then show that $\bigoplus_{j\in I} \mathrm{Hom}(R_j,R_k)$ is a slender group for all $k\in I$. Applying (2.3)(ii) we get

$\mathrm{Hom}(V,\mathrm{Hom}(V,V)) \cong \prod_{k\in I} \bigoplus_{i\in I} \mathrm{Hom}(R_i, \bigoplus_{j\in I} \mathrm{Hom}(R_j,R_k))$. Hence

$\mathrm{Hom}(R_i, \bigoplus_{j\in I} \mathrm{Hom}(R_j,R_k)) \neq 0$ for some i and $k \in I$, so from Lemma 2.4 we conclude that $T(R_i)\ T(R_j) \leqq T(R_k)$ for some $j \in I$.

Corollary 2.6. Let $V = \prod_{i\in I} R_i$ be a vector group, where I is not measurable. Then V is nil if and only if $\bigoplus_{i\in I} R_i$ is nil.

We now turn our attention to the absolute annihilator $V(*)$ of a vector group V.

Theorem 2.7. Let $V = \prod_{i\in I} R_i$ be a vector group with the index set I not measurable, and let

$I_1 = \{i \in I \mid$ there exist no j and $k \in I$ with $T(R_i)\ T(R_j) \leqq$
$\leqq T(R_k)\}$.

Then $V(*) = \prod_{i\in I_1} R_i$.

Proof: Let $v \in V(*)$. Write $v = (\ldots, r_i, \ldots)$ where some $r_i \neq 0$, $r_i \in R_i$ and assume there exist j, $k \in I$ with $T(R_i)\ T(R_k) \leqq$ $\leqq T(R_k)$. Applying Proposition 1.1 we obtain an associative ring $(\mathcal{R}', \times')$ on a finite rank completely decomposable summand $V_0 = \bigoplus_{i_0 \in I_0} R_{i_0}$ of V, $V = V_0 \oplus V'$, such that $i \in I_0$, $R_i \times' R_\ell \neq$ $\neq 0$ for some $\ell \in I_0$ and $R_m \times' R_\ell = 0$ for all $m \in I_0$, $m \neq i$. We can extend $(\mathcal{R}', \times')$ to a ring $(\mathcal{R}, \times)$ on V by letting $\times$ coincide with $\times'$ on V_0, and letting all other products be zero. Now $v = \sum_{i_0 \in I_0} r_{i_0} + v'$ where $v' \in V'$. Thus $0 = v \times r_\ell = r_i \times$ $\times r_\ell$ for all $r_\ell \in R$. This cannot be the case since $R_i \times'$ $\times' R_\ell \neq 0$, whence $v \in \prod_{i\in I_1} R_i$.

Conversely, suppose $v \in \prod_{i\in I_1} R_i$. If R_j is divisible for

some $j \in I$ then I_1 is empty. so $v = 0$ and so $v \in V(*)$. Hence R_j can be assumed to be reduced for all $j \in I$. Write $v = (\ldots, r_i, \ldots)$ where some $r_i \neq 0$, $r_i \in R_i$. Suppose $v \notin V(*)$. Then there is a $\phi \in \mathrm{Hom}(V, \mathrm{Hom}(V,V))$ with $\phi(v) \neq 0$. Thus $\mathrm{Hom}(\prod_{i \in I_1} R_i, \mathrm{Hp},(V,V)) \neq 0$. (2.1), (2.2) and (2.3)(ii) imply $\mathrm{Hom}(\prod_{i \in I_1} R_i, \mathrm{Hom}(\prod_{j \in I} R_j, \prod_{k \in I} R_k)) \cong$
$\cong \prod_{k \in I} \bigoplus_{i \in I_1} \mathrm{Hom}(R_i, \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k))$, so there is an $i \in I_1$ and $k \in I$ with $\mathrm{Hom}(R_i, \bigoplus_{j \in I} \mathrm{Hom}(R_j, R_k)) \neq 0$. From Lemma 2.4 we infer that $T(R_i)\ T(R_j) \leqq T(R_k)$ for some $j \in I$, contrary to our choice of v. Hence v is in $V(*)$.

Consider the chain

$$0 \leq V(1) \leq V(2) \leq \ldots \leq V(\alpha) \leq \ldots$$

of subgroups of V defined inductively as follows:

$V(1) = V(*)$; $V(\alpha+1)/V(\alpha) = [V/V(\alpha)](*)$; $V(\beta) = \bigcup_{\alpha<\beta} V(\alpha)$ if β is a limit ordinal. It is clear that $V(\mu+1) = V(\mu)$ for some ordinal μ.

As in [4] we introduce π-matrices in order to give a description of $V(n)$ for n finite. A $2 \times m$ π <u>-matrix</u> is a $2 \times m$ matrix of types

$$\begin{bmatrix} \tau_{11} & \tau_{12} \ldots & \tau_{1m} \\ \tau_{21} & \tau_{22} \ldots & \tau_{2m} \end{bmatrix}$$

such that $\tau_{1i}\ \tau_{2i} \not\leq \tau_{1i+1}$ for $i = 1,2,\ldots,m-1$.

<u>Proposition 2.8.</u> Let $V = \prod_{i \in I} R_i$ be a vector group with I not measurable, and for each positive integer n let $I_n = \{i \in I \mid$ there exists no $2 \times (n+1)$ π-matrix over $\{T(R_j) \mid j \in I\}$ with $\tau_{11} = T(R_i)\}$. Then $V(n) = \prod_{i \in I_n} R_i$.

Proof: See the proof of Proposition 2.5 of [4].

We then have

<u>Theorem 2.9.</u> Let $V = \prod_{i \in I} R_i$ be a vector group with the index set I not measurable. Then the following conditions are equivalent:

(1) $V = V(n)$, $n < \infty$ and $V \neq V(n-1)$;

(2) there are $2 \times n$, but no $2 \times (n+1)$ π-matrices over $\{T(R_i) \mid i \in I\}$;

(3) V has strong nil-degree n.

Proof: See the proof of Theorem 4.2 of [4].

<u>Corollary 2.10.</u> Let $V = \prod_{i \in I} R_i$ be a vector group with I not measurable. Then V and $\bigoplus_{i \in I} R_i$ have the same strong nil-degree.

Proof: Theorem 4.2 of [4] shows that Theorem 2.9 is true when $V = \prod_{i \in I} R_i$ is replaced by $\bigoplus_{i \in I} R_i$.

We conclude this section with some necessary conditions for a direct product of slender groups to be nil.

<u>Proposition 2.11.</u> Let $A = \prod_{i \in I} A_i$, where the A_i are slender and the index set I is not measurable, $(\mathcal{R}, \times)$ a ring on A. If $\bigoplus_{i \in I} A_i$ is a subgroup of $(0: (\mathcal{R}, \times))$ then $(\mathcal{R}, \times)$ is the trivial ring on A.

Proof: Let $\phi \in \mathrm{Hom}(\prod_{i \in I} A_i, \mathrm{Hom}(\prod_{j \in I} A_j, \prod_{k \in I} A_k))$ be the map defining $(\mathcal{R}, \times)$ (thus $\phi(a)b = a \times b$ for all $a, b \in A$). Under the natural isomorphism $\mathrm{Hom}(\prod_{j \in I} A_j, \prod_{k \in I} A_k) \cong$ $\cong \prod_{k \in I} \mathrm{Hom}(\prod_{j \in I} A_j, A_k)$, $\phi(a) \longrightarrow (\ldots, \pi_k \phi(a), \ldots)$, where $\pi_k : \prod_{i \in I} A_i \longrightarrow A_k$ is the projection, for all $k \in I$. Now for each $a' \in \bigoplus_{i \in I} A_i$ we have $\pi_k \phi(a)a' = \pi_k(a \times a') = 0$ for all k, so (2.3)(i) implies that $\pi_k \phi(a) = 0$ for all $k \in I$ and all $a \in A$. Thus $\phi(a) = 0$ for all $a \in A$, i.e. $a \times b = 0$ for all $a, b \in A$.

Corollary 2.12. Let $A = \prod_{i \in I} A_i$ be a direct product of slender groups where I is not measurable. If $\bigoplus_{i \in I} A_i$ is a subgroup of $A(*)$, then A is nil.

We need the following result.

Lemma 2.13. Let $\{A_n \mid n = 1,2,\ldots\}$ be a countable family of torsion-free groups, and B be an arbitrary group. If $\mathrm{Hom}(\bigoplus_{n=1}^{\infty} A_n, B) = 0$ then $\mathrm{Hom}(\prod_{n=1}^{\infty} A_n, B) = 0$.

Proof: See Proposition 7.3 of [3].

Proposition 2.14. Let $A = \prod_{n=1}^{\infty} A_n$ be a countable direct product of slender groups such that $\bigoplus_{n=1}^{\infty} A_n$ is nil. Then A is nil.

Proof: Observe that since each A_n is slender, (2.3)(i) implies that $\mathrm{Hom}(\prod_{m=1}^{\infty} A_m / \bigoplus_{m=1}^{\infty} A_m, A_n) = 0$ for all n, so applying $\mathrm{Hom}(\prod_{k=1}^{\infty} A_k, \circ)$ to the exact sequence

$$0 = \prod_{n=1}^{\infty} \mathrm{Hom}(\prod_{m=1}^{\infty} A_m / \bigoplus_{m=1}^{\infty} A_m, A_n) \cong$$

$$\cong \mathrm{Hom}(\prod_{m=1}^{\infty} A_m / \bigoplus_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n) \longrightarrow \mathrm{Hom}(\prod_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n),$$

we see that A is nil if $\mathrm{Hom}(\prod_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n)) = 0$. Now $\bigoplus_{k=1}^{\infty} A_k$ is nil, so $\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, \bigoplus_{n=1}^{\infty} A_n)) = 0$, whence $\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, A_n)) = 0$ for all n, so

$\mathrm{Hom}(\bigoplus_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n)) = 0$. By Lemma 2.13, we then have $\mathrm{Hom}(\prod_{k=1}^{\infty} A_k, \mathrm{Hom}(\bigoplus_{m=1}^{\infty} A_m, \prod_{n=1}^{\infty} A_n)) = 0$, so A is nil.

3. Separable groups. A torsion-free group A is called separable if every finite set elements of A is contained in a completely decomposable direct summand of A. It is clear that we can choose this summand with finite rank.

We commence this section with a description of the nil separable groups. First, however, we need to consider the following subgroups of a separable group.

Suppose $(\mathcal{R}, \times)$ is a ring on the separable group A, and $A_1 \oplus A_2$ is a finite rank completely decomposable direct summand of A. We are permitted to write $A_1 = \langle a_1 \rangle_* \oplus \langle a_2 \rangle_* \oplus \cdots \oplus \langle a_{n_1} \rangle_*$ and $A_2 = \langle a_{n_1+1} \rangle_* \oplus \langle a_{n_1+2} \rangle_* \oplus \cdots \oplus \langle a_{n_2} \rangle_*$ for suitable elements $a_1, a_2, \ldots, a_{n_2}$ of A, and $A = A_1 \oplus A_2 \oplus A_2'$ for some subgroup A_2' of A. Since A_2' is a direct summand of A, Theorem 87.5 of [2] shows it is separable, and so there is a finite rank completely decomposable direct summand A_3 of A_2' with the property that $A_1 \oplus A_2 \oplus A_3$ contains all products of the form $a_i \times a_j$ where $i \in \{1, 2, \ldots, n_1\}$ and $j \in \{1, 2, \ldots, n_2\}$. Thus $A_3 = \langle a_{n_2+1} \rangle_* \oplus \langle a_{n_2+2} \rangle_* \oplus \cdots \oplus \langle a_{n_3} \rangle_*$ for suitable elements $a_{n_2+1}, a_{n_2+2}, \ldots, a_{n_3}$ of A. Since $A_1 \oplus A_2 \oplus A_3$ is a pure subgroup of A it is clear that $a \times b \in A_1 \oplus A_2 \oplus A_3$ for all $a \in A_1$ and all $b \in A_1 \oplus A_2$.

Lemma 3.1. Let $(\mathcal{R}, \times)$ be a ring on a separable group A, and let A_1, A_2, and A_3 be subgroups of A defined as above.

If $\operatorname{Hom}(A_1, \operatorname{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \neq 0$ then there exist $i \in \{1,2,\ldots,n_1\}$, $j \in \{1,2,\ldots,n_2\}$ and $k \in \{1,2,\ldots,n_3\}$ such that $T(a_i)\ T(a_j) \leq T(a_k)$.

Proof: Clearly

$$\operatorname{Hom}(A_1, \operatorname{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \cong$$

$$\cong \bigoplus_{i=1}^{n_1} \bigoplus_{j=1}^{n_2} \bigoplus_{k=1}^{n_3} \operatorname{Hom}(\langle a_i \rangle_* , \operatorname{Hom}(\langle a_j \rangle_* , \langle a_k \rangle_*)).$$

Proceeding as in the proof of Lemma 2.4 we obtain the required result.

Theorem 3.2. Let A be a separable group. Then the following conditions are equivalent:

(1) A is strongly nil;

(2) A is nil;

(3) every rank n ($n \leq 3$) completely decomposable direct summand of A is nil.

Proof: Clearly (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3). It remains to show (3) $\Longrightarrow$ (1). Suppose there is a ring $(\mathcal{R}, \times)$ on A, and elements $a, b \in A$ with $a \times b \neq 0$. Let A_1 be a finite rank completely decomposable direct summand of A containing a and b, and let $A_2 = 0$. Define A_3 as we did prior to Lemma 3.1. For $e \in A_1$ define $\phi : A_1 \longrightarrow \operatorname{Hom}(A_1, A_1 \oplus A_3)$ by $\phi(e)f = e \times f$ for all $f \in A_1$. Then $\phi \in \operatorname{Hom}(A_1, \operatorname{Hom}(A_1, A_1 \oplus A_3))$ and $\phi(a)b = a \times b \neq 0$. We now apply Lemma 3.1 and Proposition 1.1. to obtain a rank n ($n \leq 3$) direct summand of A which is non-nil.

We now turn our attention to the absolute annihilator $A(*)$ of a separable group A. We need to make the following definitions.

A finite set of elements $\{a_1, \ldots, a_n\}$ of a separable group A is called basic if it is linearly independent and $\langle a_1 \rangle_* \oplus \langle a_2 \rangle_* \oplus \ldots \oplus \langle a_n \rangle_*$ is a direct summand of A. An element $a \in A$ is a basic element of A if the set $\{a\}$ is basic. For a separable group A we define

$A' = \{a \in A \mid a$ is a basic element of A with the property that there do not exist basic elements $b, c \in A$ with $\{a, b, c\}$ basic and $T(a)\ T(b) \leqq T(c)\}$.

Proposition 3.3. Let A be a separable group and let A' be defined as above. Then $A(*)$ is the pure subgroup of A generated by A'.

Proof: If $a \in \langle A' \rangle_*$ then we can write $na = n_1a_1 + n_2a_2 + \ldots + n_ka_k$ where $n, n_1, n_2, \ldots, n_k$ are integers and $a_i \in A'$ for $i = 1, 2, \ldots, k$. If $a_i \notin A(*)$ for some $i \in \{1, 2, \ldots, k\}$ then there is a ring $(\mathcal{R}, \times)$ on A with $a_i \times a \neq 0$ for some $a \in A$. Let $A_1 = \langle a_i \rangle_*$, and $A_2 = \langle a_2 \rangle_* \oplus \langle a_3 \rangle_* \oplus \ldots \oplus \langle a_{n_2} \rangle_*$ be such that $A_1 \oplus A_2$ is a completely decomposable summand of A containing a. Define A_3 as we did prior to Lemma 3.1. As in the proof of Theorem 3.2, $a_i \times a \neq 0$ implies that $\mathrm{Hom}(A_1, \mathrm{Hom}(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \neq 0$, so Lemma 3.1 shows that $T(a_i)\ T(a_j) \leqq T(a_k)$ for some $j \in \{i, 2, 3, \ldots, n_2\}$ and $k \in \{i, 2, 3, \ldots, n_3\}$, which contradicts our assumption that $a_i \in A'$. Hence each a_i is in $A(*)$, so $na \in A(*)$, and since $A(*)$ is pure in A it follows that $a \in A(*)$.

Conversely, suppose $a \in A(*)$. Now a can be embedded in a finite rank completely decomposable direct summand A_1 of A, $A_1 = \langle a_1 \rangle_* \oplus \langle a_2 \rangle_* \oplus \ldots \oplus \langle a_{n_1} \rangle_*$, and there exist integers $n, n_1, n_2, \ldots, n_{n_1}$ such that $na = n_1a_1 + n_2a_2 + \ldots$

$\ldots + n_{n_1} a_{b_1}$. If $a_i \notin A'$ for some $i \in \{1,2,\ldots n_1\}$ then there are basic elements $b, c \in A$ such that $\{a_i, b, c\}$ is basic and $T(a_i)\ T(b) \leqq T(c)$. By Proposition 1.1 there exists a ring $(\mathcal{R}, \times)$ on A with $a_i \times a' \neq 0$ for some $a' \in A$. If we let
$A_2 = \langle a_{n_1+1} \rangle_* \oplus \langle a_{n_1+2} \rangle_* \oplus \quad \ldots \oplus \langle a_{n_2} \rangle_*$
be such that $A_1 \oplus A_2$ is a completely decomposable summand of A containing a', and define A_3 as usual, then as in the proof of Theorem 3.2, $a_i \propto a' \neq 0$ implies that
$Hom(\langle a_i \rangle_*, Hom(A_1 \oplus A_2, A_1 \oplus A_2 \oplus A_3)) \neq 0$. Applying Lemma 3.1 we see that $T(a_i)\ T(a_j) \leqq T(a_k)$ for some $j \in \{1,2,\ldots,n_2\}$ and $k \in \{1,2,\ldots,n_3\}$. Proposition 1.1 then shows that we can define a ring $(\mathcal{R}', \times')$ on $A_1 \oplus A_2 \oplus A_3$ with
$\langle a_i \rangle_* \times' \langle a_\ell \rangle_* \neq 0$ for some $\ell \in \{1,2,\ldots,n_3\}$ and
$\langle a_m \rangle_* \times' \langle a_\ell \rangle_* = 0$ for all $m \in \{1,2,\ldots,n_3\}$, $m \neq i$. We can extend $\times'$ to A by setting all other products equal to 0. But then $0 = (na) \times' a_\ell = (n_i a_i) \times' a_\ell$. We conclude that $a \in \langle A' \rangle_*$.

We end with some results concerning the absolute annihilator series of an arbitrary torsion-free group. Recall that for a torsion-free group A, this is defined inductively as follows: $A(1) = A(*)$, $A(\alpha + 1)/A(\alpha) = [A/A(\alpha)](*)$ and $A(\beta) = \bigcup_{\alpha<\beta} A(\alpha)$ if β is a limit ordinal.

<u>Proposition 3.4.</u> Let A be a torsion-free group and $(\mathcal{R}, \times)$ a ring on A. Then $A(\alpha)$ is an ideal in $(\mathcal{R}, \times)$ for all ordinals α.

Proof: First we show $A(*)$ to be fully invariant in A.

Let f be in Hom(A,A) and $a \in A$. If $f(a) \notin A(*)$ then there is a homomorphism $\phi \in$ Hom(A,Hom(A,A)) with $\phi(f(a)) \neq 0$. But $\phi f \in$ Hom(A,A)) and $(\phi f)(a) \neq 0$, so $a \notin A(*)$.

A transfinite induction argument shows that $A(\alpha)$ is fully invariant in A for all ordinals α . The result now follows immediately.

Corollary 3.5. If $A = A(\mu)$ for some ordinal μ then any associative ring $(\mathcal{R}, \mu)$ on A is left and right T-nilpotent. If in addition μ is finite, then $(\mathcal{R}, \times)^{\mu+1} = 0$.

Proof: See the proof of Corollary 2.4 of [4].

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