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K-ESSENTIAL SUBGROUPS OF ABELIAN GROUPS . I

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Abstract: The purpose of this paper is to investigate K-essential subgroups of abelian groups and to generalize some known results about essential subgroups. In the section 1 there are presented some fundamental properties of the K-essential subgroups. In the section 2 there are given the necessary and sufficient conditions for the existence of a (maximal) K-essential extension. The section 3 investigates sets of K-essential subgroups, where K runs through the subgroups of a group G.

Key words: K-essential, essential subgroups; (maximal) K-essential extensions; N-high, N-K-high subgroups.

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0. **Introduction.** All groups considered here are abelian. Concerning the terminology and notation, we refer to [1] and [2]. Otherwise, if G is a group then G_t and G_p are the torsion part and p-component of G_t respectively.

We shall frequently use the following notation:

$\mathbb{Q}$ - group of rationals,

$\mathbb{Z}$ - group of integers, infinite cyclic group,

$\mathbb{Z}(n)$ - cyclic group of order n,

$\mathbb{N} = \{n \in \mathbb{Z} \ ; \ n > 0\}$,

$\mathbb{P}$ - the set of all prime numbers,

(n,m) - the greatest common divisor of n and m,

$h_p^G(g)$ - the p-height of g in G.

Let $K \subset N$ be subgroups of a group G. Following Krivonos [3], a subgroup A of G is said to be N-K-high in G if A is maximal with respect to the property $A \cap N = K$.

1. K-essential subgroups

Definition 1.1. Let G be a group and K a subgroup of G. A subgroup N of G is said to be K-essential in G if for every $g \in G \setminus K$ there is $n \in N$ such that $ng \in N \setminus K$.

Remark 1.2. 0-essential subgroups of a group G are exactly the essential subgroups of G and every subgroup of G is G-essential in G. If K is a proper subgroup of G then no subgroup of K is K-essential in C. The group G is K-essential in G for every subgroup K of G. If K is a proper subgroup of a mixed group G then no torsion subgroup of G is K-essential in G.

The proof of the following proposition is straightforward and hence omitted.

Proposition 1.3. Let N and K be subgroups of a group G. Then the following are equivalent:

(i) N is K-essential in G;

(ii) K is the unique $N - N \cap K$-high subgroup of G;

(iii) if $\alpha : G \to A$ is a homomorphism and $\mathrm{Ker}(\alpha | N) \subset K$ then $\mathrm{Ker}\, \alpha \subset K$;

(iv) if X is a subgroup of G and $X \not\subset K$ then $X \cap N \not\subset K$;

(v) if H is a subgroup of G then $N \cap H$ is $K \cap H$-essential in H;

(vi) if $g \in G$ then $N \cap \langle g \rangle$ is $K \cap \langle g \rangle$-essential in $\langle g \rangle$;

(vii) if L is a subgroup of K then $N+L/L$ is K/L-essential in G/L.

If N is a K-essential subgroup of a group G and A is a subgroup of G with $A \cap K = 0$ then $A \cap N$ is essential in A by 1.3 (v).

If K and N are subgroups of a group G then K is $N - N \cap \cap K$-high in G iff $N + K$ is K-essential in G.

Proposition 1.4. The family of all K-essential subgroups of a group G is a filter.

Proof. If N is a K-essential subgroup of G and M a subgroup of G containing N then M is K-essential in G. If N and M are K-essential subgroups of G then $N \cap M$ is K-essential in G.

Proposition 1.5. Let G be a group. Then

(i) If N is a K_i-essential subgroup of G for every $i \in I$ then N is $\bigcap K_i$-essential in G.

(ii) If N is a K_i-essential subgroup of G for every $i \in I$, where $\{K_i; i \in I\}$ is a chain of subgroups of G, then N is $\bigcup K_i$-essential in G.

(iii) If N is a K-essential subgroup of H, where H is a K-essential subgroup of G, then N is K-essential in G.

Proof. (i) If $g \in G \setminus \bigcap K_i$ then there is an element $j \in I$ such that $g \in G \setminus K_j$. Now, there is $n \in \mathbb{N}$ with $ng \in N \setminus K_j$ and hence $ng \in N \setminus \bigcap K_i$.

(ii) Let $g \in G \setminus \bigcup K_i$. For every $i \in I$ there is $n_i \in \mathbb{N}$ such that $n_i g \in N \setminus K_i$. If $\langle g \rangle \cap N = \langle ng \rangle$ then obviously $ng \in \in N \setminus K_i$ for every $i \in I$.

(iii) If $g \in G \setminus K$ then there is $n \in \mathbb{N}$ such that $ng \in H \setminus K$. Further, there is $m \in \mathbb{N}$ with $mng \in N \setminus K$.

Remark 1.6. Let K_i and N_i be subgroups of a group G, $i \in I$.

If $\oplus N_i$ is $\oplus K_i$-essential in $\oplus G_i$ then N_i is K_i-essential in G_i for every $i \in I$. Obviously, the same applies to direct products.

Proposition 1.7. Let $K_i \subset N_i$ be subgroups of a group G_i, $i \in I$. Then $\oplus N_i$ is $\oplus K_i$-essential in $\oplus G_i$ iff N_i is K_i-essential in G_i for every $i \in I$.

Proof. Let $g \in \oplus G_i \setminus \oplus K_i$; i.e. $g = \sum_{j=1}^{m} g_{i_j}$, where $g_{i_j} \in G_{i_j}$ for every $j = 1,\dots,m$, $m \in \mathbb{N}$ and there is an integer k, $1 \leq k \leq m$, such that $g_{i_k} \in G_{i_k} \setminus K_{i_k}$. Since N_{i_k} is K_{i_k}-essential in G_{i_k}, there is $n \in \mathbb{N}$ such that $ng_{i_k} \in N_{i_k} \setminus K_{i_k}$. If $ng \in \oplus N_i$ then we are through, since obviously $ng \notin \oplus K_i$. If $ng \notin \oplus N_i$ then there is an element $r \in \mathbb{N}$, $1 \leq r \leq m$ such that $ng_{i_r} \in G_{i_r} \setminus K_{i_r}$ and so on. The converse follows from 1.6.

If G is a group and K is a torsion subgroup of G, then the torsion parts of all the K-essential subgroups of G are K-essential in G_t by 1.3. The following proposition implies that all the K-essential subgroups in G_t can be obtained in this very way.

Proposition 1.8. If K is a torsion subgroup of a group G and L is a K-essential subgroup of G_t then every G_t-L-high subgroup of G is K-essential in G.

Proof. Let N be a G_t-L-high subgroup of G and $g \in G \setminus K \cup N$. If $g \in G_t$ then there is $n \in \mathbb{N}$ such that $ng \in L \setminus K \subset N \setminus K$. If $g \notin G_t$ then $\langle g,N \rangle_t \supsetneqq L$; there are $k \in \mathbb{N}$, $m \in N$ and $t \in G_t \setminus L$ such that $kg + m = t$. Now, $\sigma(t)kg \in N \setminus K$.

Proposition 1.9. Let N be a K-essential subgroup of a group G and A be a subgroup of G. Then A is N-high in G iff A is $N \cap K$-high in K.

Proof. If A is N-high in G then A is a subgroup of K by 1.1.

Proposition 1.10. Let k, n be nonnegative integers. Then the subgroup $n\mathbb{Z}$ is $k\mathbb{Z}$-essential in $\mathbb{Z}$ iff $h_p^{\mathbb{Z}}(k) \geq 1$ implies $h_p^{\mathbb{Z}}(n) < h_p^{\mathbb{Z}}(k)$.

Proof. The assertion is obviously for k = 0. Suppose $k \in \mathbb{N}$.

Let $n\mathbb{Z}$ be $k\mathbb{Z}$-essential in $\mathbb{Z}$ and k = pr, where $p \in \mathbb{P}$ and $r \in \mathbb{N}$. Obviously $r \notin k\mathbb{Z}$; consequently there is $m \in \mathbb{N}$ such that $mr \in n\mathbb{Z} \setminus k\mathbb{Z}$. Hence $n \mid mr$ and $(p,m) = 1$. If $p^i \mid n$ then $p^i \mid r$ and $p^{i+1} \mid k$.

Conversely, let $x \notin k\mathbb{Z}$. Now, there are $p \in \mathbb{P}$ and $i \in \mathbb{N}$ such that $h_p^{\mathbb{Z}}(k) = i$ and $h_p^{\mathbb{Z}}(x) < i$. We can write x = ya, n = yb, where (a,b) = 1. Obviously $bx = an \in n\mathbb{Z}$. If $bx \in k\mathbb{Z}$ then $p^i \mid bx$. Hence $p \mid b$ and $(p,a) = 1$. Consequently, $p^i \mid n$ and $p^{i+1} \mid k$, a contradiction. Hence $bx \in n\mathbb{Z} \setminus k\mathbb{Z}$.

2. Maximal K-essential extensions

Definition 2.1. Let N and K be subgroups of a group G. The group G is said to be a K-essential extension of N, if N is K-essential in G. The group G is said to be a maximal K-essential extension of N, if N is K-essential in G and is not K-essential in a group H, whenever G is a proper subgroup of H.

Let G be a maximal K-essential extension of a group N. Then G is a maximal K-essential extension of a subgroup M of N iff M is $K \cap N$-essential in N.

Theorem 2.2. Let N and K be subgroups of a group A. Then the following are equivalent:

(i) There exists a K-essential extension of N;

(ii) There exists a maximal K-essential extension of N;

(iii) N is K-essential in N + K;

(iv) Either $K/K\cap N$ is a torsion group and $(K/K\cap N)_p \neq 0$ implies $(N/K\cap N)_p = 0$ or $N \subset K$.

Proof. (i)$\Longrightarrow$(iii). If G is a K-essential extension of the group N then N is K-essential in N + K by 1.3 (v).

(iii)$\Longrightarrow$(ii). Let D be a divisible group containing A and $\mathfrak{M}$ be the set of all K-essential extensions of N, that are contained in the group D. It is $N + K \in \mathfrak{M}$ by (iii), $\mathfrak{M}$ is partially ordered and inductive. By Zorn's lemma, there is a maximal element G in $\mathfrak{M}$. Suppose N is K-essential in a group H, where $G \subset H$. Now, there exists a homomorphism $f: H \longrightarrow D$ extending the natural inclusion G into D. Consequently, $\mathrm{Ker}(f \mid N) = 0 \subset K$ and by 1.3 (iii), $\mathrm{Ker}\, f \subset K$; hence f is a monomorphism. If $g \in f(H) \setminus K$ then $f^{-1}(g) \in H \setminus K$ and there is $n \in \mathbb{N}$ such that $nf^{-1}(g) \in N \setminus K$, i.e. $ng \in N \setminus K$. Consequently, the group f(H) is a K-essential extension of N and $G \subset f(H) \subset D$. Hence G = H.

(ii)$\Longrightarrow$(i). Trivial.

(iii)$\Longleftrightarrow$(iv). If $m \in N$, $k \in K$ and $n \in \mathbb{N}$ then

$m + k \in N + K \setminus K$ iff $m \notin K \cap N$

and

$nm + nk \in N \setminus K$ iff $nm \notin K \cap N$ and $nk \in K \cap N$.

Moreover, if $m \in N \setminus K \cap N$ and $k \in K \cap N$ then $m + k \in N \setminus K$ (in this case n = 1). Hence, the assertion (iii) is equivalent to the assertion

(v) if $m \in N \setminus K \cap N$ and $k \in K \setminus K \cap N$ then there is $n \in \mathbb{N}$ such that $nm \notin K \cap N$ and $nk \in K \cap N$.

(iv) $\Longrightarrow$ (v). Suppose $m \in N \setminus K \cap N$ and $k \in K \setminus K \cap N$. Let n be the least nonzero natural number such that $nk \in K \cap N$. Such number exists, since $K/K \cap N$ is a torsion group by (iv). If $n = pr$, where $p \in \mathbb{P}$ and $r \in \mathbb{N}$, then $rk + K \cap N$ is a nonzero element of the group $(K/K \cap N)_p$. For each prime p with $p \mid n$ $(N/K \cap N)_p = 0$ by (iv). Hence $nm \notin K \cap N$.

(v) $\Longrightarrow$ (iv). If N is not a subgroup of K, i.e. $N \supsetneqq K \cap N$, then $K/K \cap N$ is a torsion group by (v). If $(K/K \cap N)_p \neq 0$ and $(N/K \cap N)_p \neq 0$ for some prime p then there are elements $k \in K \setminus K \cap N$ and $m \in N \setminus K \cap N$ such that $pk, pm \in \in K \cap N$. By (v), there is $n \in \mathbb{N}$ such that $nm \notin K \cap N$ and $nk \in \in K \cap N$. Now, $(p,n) = 1$ and there are integers u, v such that $up + vn = 1$. Hence $k = upk + vnk \in K \cap N$, a contradiction.

<u>Proposition 2.3.</u> Let N and K be subgroups of a group G and there exists a K-essential extension of the group N. Then

(i) There exists a subgroup of G that is maximal with respect to the property of being a K-essential extension of N im G;

(ii) The group G is a K-essential extension of N iff G/K is an essential extension of the group $N+K/K$;

(iii) The group G is a maximal K-essential extension of N iff G/K is a divisible hull of the group $N+K/K$;

(iv) If G/K is a divisible group then G contains a maximal K-essential extension of the group N.

Proof. (i) Let $\mathfrak{M}$ be the set of all K-essential extensions of the group N that are contained in G. By 2.2, the group N + K is an element of $\mathfrak{M}$. The set $\mathfrak{M}$ is partially ordered and inductive. By Zorn's lemma there is a maximal element in $\mathfrak{M}$.

(ii) Suppose that the group G/K is an essential extension of the group $N+K/K$. If $g \in G \setminus K$ then there is $n \in \mathbb{N}$ such that $ng = m + k$, where $m \in N \setminus K$ and $k \in K$. Let r be the order of the element $k + K \cap N$. By 2.2, r is finite and $rm \notin K$. Hence $rng \in N \setminus K$. The converse follows from 1.3 (vii).

(iii) It follows immediately from (ii).

(iv) The group G/K contains a divisible hull D/K of the group $N+K/K$ and hence D is a maximal K-essential extension of N by (iii).

If G is a maximal K-essential extension of N then G is an extension of K by a divisible hull of the group $N+K/K$.

Corollary 2.4. Let K and N be subgroups of a group G. Then

(i) If $N \subset K$ then K is the unique maximal K-essential extension of N;

(ii) If $K \subset N$ then there is a maximal K-essential extension of N. The group G is a maximal K-essential extension of N iff G/K is a divisible hull of N/K;

(iii) If $K \cap N = 0$ and N is nonzero then there exists a maximal K-essential extension of N iff K is torsion and $K_p \neq 0$ implies $N_p = 0$. If it holds then G is a maximal K-essential extension of N iff G/K is a divisible hull of $N \oplus K/K$.

Remark 2.5. The existence-assumption of K-essential extension of N in 2.3 cannot be omitted as it is seen from the following example: Suppose $G = N \oplus K$, where $N \cong \mathbb{Q}$ and $K \cong \mathbb{Z}$. Then the group $^{N+K}/K$ is essential in $^{G}/K$ and $^{G}/K$ is divisible. But N is not a K-essential subgroup of G.

Remark 2.6. In [4], T. Szele investigates the algebraic elements with respect to a given subgroup and problem of algebraic extensions of groups. It is closely related to the problem of K-essential subgroups as it follows.

Let N and K be subgroups of a group G. An element g of G is said to be K-algebraic over N if $g \in K$ or $\langle g \rangle \cap N \cap K \neq \langle g \rangle \cap N$. If every $g \in G$ is K-algebraic over N, we call G K-algebraic over N. In this case the group G is also said to be K-algebraic extension of N. Now, it is easy to see that G is K-algebraic over N iff N is K-essential in G. Hence, K-essential extensions of N and K-algebraic extensions of N are the same.

3. Comparing K-essential subgroups

Lemma 3.1. Let G be a group and N a subgroup of G such that $^{G}/N$ is a torsion group. Let K be a subgroup of G such that if $(^{G}/N)_p \neq 0$ then $G_p \subset K$ and K is p-pure in G. Then N is K-essential in G.

Proof. Let $g \in G \setminus K$. If $n = o(g + N)$ then $ng \in N$. Moreover, $ng \notin K$. For, if $ng \in K$ then there is $k \in K$ such that $ng = nk$. Now, $n(g - k) = o$ and consequently $g - k \in K$. Hence $g \in K$ implies a contradiction.

<u>Lemma 3.2.</u> Let G be a group. If K is a proper subgroup and N is a K-essential subgroup of G then ${}^{G}/N$ is a torsion group.

<u>Proof.</u> Suppose $g + N \in {}^{G}/N$. If $g \notin K$ then there exists $n \in \mathbb{N}$ such that $ng \in N \setminus K$. If $g \in K$ then by 2.2 either $N \subset K$ (hence K = G, a contradiction) or ${}^{K}/K \cap N$ is a torsion group. Consequently, there is $m \in \mathbb{N}$ such that $mg \in N$.

<u>Theorem 3.3.</u> Let G be a group with subgroups K and N. Then

(i) N is G_t-essential in G iff ${}^{G}/N$ is torsion;

(ii) If K is proper and N is K-essential in G then N is G_t-essential in G;

(iii) If G is not torsion then the set of all K-essential subgroups of G and the set of all G_t-essential subgroups of G are identical iff K is proper pure containing G_t;

(iv) If K is proper then each K-essential subgroup of G is K_t-essential in G;

(v) If K is proper and torsion-free then each K-essential subgroup of G is essential in G.

<u>Proof.</u> (i) Let N be G_t-essential in G. If $G = G_t$, then ${}^{G}/N$ is torsion. If $G \neq G_t$ then ${}^{G}/N$ is torsion by 3.2. Conversely, if ${}^{G}/N$ is a torsion group then N is a G_t-essential subgroup of G by 3.1.

(ii) It follows immediately from 3.2 and (i).

(iii) Let K be a proper pure subgroup of G containing G_t. If N is G_t-essential in G then ${}^{G}/N$ is a torsion group by (i) and N is K-essential in G by 3.1. The converse follows from (ii).

Suppose that the set of all G_t-essential subgroups of G and the set of all K-essential subgroups of G are identical. If G_t is not a subgroup of K then each G_t-high subgroup of G is G_t-essential in G and not K-essential in G. Next, we suppose that K contains G_t and K is not pure in G. There exists $g \in G \setminus K$ such that $\langle g \rangle \cap K = \langle kg \rangle$, where $k \in \mathbb{N}$. Let N be a subgroup maximal with respect to the properties: $N_t = 0$, $N \cap \langle g \rangle = \langle kg \rangle$. Obviously, N is not K-essential in G. If $x \in G \setminus G_t \cup N$ then either $\langle x,N \rangle_t \neq 0$ or $\langle x,N \rangle \cap$ $\cap \langle g \rangle \supsetneqq \langle kg \rangle$. In the first case there are $n \in \mathbb{N}$, $h \in N$ and $t \in G_t$ such that $nx + h = t$; hence $\sigma(t)nx \in N \setminus G_t$. In the second case there are $m,n \in \mathbb{N}$ and $h \in N$ such that $nx + h =$ $= mg$. Now, $kmg = knx + kh$, i.e. $knx \in N \setminus G_t$. Consequently N is G_t-essential in G, a contradiction. Hence K is a pure subgroup of G containing G_t, K is proper since G is not torsion.

(iv) If N is a K-essential subgroup of G then N is G_t-essential in G by (ii) and N is K_t-essential in G by 1.5.

(v) It follows from (iv).

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