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FUZZY MAPPINGS AND FUZZY SETS

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Abstract: It is shown that in the language of fuzzy sets various notions of dispersed mappings (more generally, dispersed morphisms associated with a category) can be represented. Moreover, this point of view is, in a sense, finer than the classical approach. - Adding the dispersed morphisms one obtains a $\mathcal{V}$-category over $\mathcal{V}$ a closed category of fuzzy sets. The $\mathcal{V}$-categories obtained in such a way are characterized.

Key words: Fuzzy (dispersed) mappings, fuzzy sets, $\mathcal{V}$-categories.

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The expression "fuzzy mappings" is loosely used for various generalizations of the notion of a mapping, in particular for those where the value in a point is in that or other way indetermined (multivalued mappings, stochastic mappings, etc.). On the other hand, in the expression "fuzzy set" the attribute indicates the possibility of incompletely present elements. Thus, these two usages of the word fuzzy appear quite incoherent: A mapping $f: X \longrightarrow Y$ is a particular kind of a subset of $X \times Y$; the question how far an $R \subset X \times Y$ is from being a mapping, how fuzzy it is in the first sense, is quite independent on the question how fuzzy it is in the second one: R can be multivalued but crisp, and on the other hand there may be for every x just one (x,y) in R, but often with an incomplete membership.

In this paper we want to show that, still, there is a way to express the fuzziness in the first sense in the language of fuzzy sets. Moreover, unlike in the classical description, the degree of fuzziness, not just the fact that it is fuzzy, is expressed.

The main idea goes as follows: Mappings between fuzzy sets are classified according to the degree in which they weaken the membership (in what extent it can happen that $f(x)$ is a weaker member of Y than x has been of X). As it is usually done in definitions of fuzzy mappings, we extend the sets (or, more generally, objects of categories) adding the possible "irregular values" (subsets, probability fields etc., see e.g. [1]), but not in the full membership. The crisp part of the extended object is still the original set (object), and the dispersedness of the new mappings is measured, roughly speaking, by the degree in which the values in the original members differ from such.

In this way, starting with a concrete category, one gets a $\mathcal{V}$-category over $\mathcal{V}$ a closed category of fuzzy sets, the crisp part of which is the original one. In the second part of this paper we show a one-to-one correspondence between dispersion procedures and a special kind (of which we present a simple characteristics) of such $\mathcal{V}$-extensions of concrete categories.

§ 1. Preliminaries

1.1. Throughout this note, L is a lattice with a least and a largest element o, e respectively. A fuzzy set X (more exactly, an L-fuzzy set) is a mapping

$$X: ?X \longrightarrow L$$

where ?X is a set.

We write

$$x \in_a X \quad \text{for} \quad X(x) \geq a.$$

Let X, Y be fuzzy sets. A morphism

$$f: X \longrightarrow Y$$

is a mapping $f: ?X \longrightarrow ?Y$ such that for every $x \in ?X$, $Y(f(x)) \geq \geq X(x)$. Thus, in the convention above, $f: ?X \longrightarrow ?Y$ is a morphism $f: X \longrightarrow Y$ iff

for every $a \in L$, $x \in_a X$ implies $f(x) \in_a Y$.

Fuzzy sets and their morphisms form a category (cf.[5]) which will be denoted by

L-Fuzz.

Associating with a fuzzy set X the set ?X and with a morphism $X \longrightarrow Y$ the corresponding mapping $?X \longrightarrow ?Y$ we obtain a (faithful) functor

$$? : \text{L-Fuzz} \longrightarrow \text{Set}$$

(Set designates the category of all sets and mappings). Further, we define a functor

$$! : \text{L-Fuzz} \longrightarrow \text{Set}$$

putting $!X = \{x \mid x \in_e X\}$ and taking for $!f$ the domain-range restriction of f.

1.2. A tensor product on L is an order-preserving semigroup operation $\square$ with unit e such that there is a homomorphism $h: L^{op} \times L \longrightarrow L$ satisfying the condition

$$a \square b \leq c \quad \text{iff} \quad a \leq h(b,c).$$

(If L is complete, a necessary and sufficient condition for the existence of such an h is that all $(a\square -)$ and $(-\square a)$ are suprema preserving mappings $L \longrightarrow L$. Thus, e.g. if L is the unit interval, the continuity of the operation $\square$ is more than sufficient.)

The couple $(L, \square)$ will be referred to as <u>tensored lattice</u> (thus, if L is complete, this notion coincides with the notion of an integral CL-monoid from [2]).

1.3. In [6] there was shown that the closedness structures

$(\otimes, H, \ldots)$ (i.e. structures of a symmetric monoidal closed category, see [4]) on L-Fuzz such that

$$?H(X,Y) = ?Y^{?X} \text{ and } H(X,Y)(f) = e \text{ for } f\colon X \longrightarrow Y$$

(i.e. such that all the mappings are in some extent members of $H(X,Y)$, the morphisms having the strongest membership possible; by the formula below it follows that then, moreover, if $H(X,Y)(f) = e$ necessarily $f\colon X \longrightarrow Y$) are in a one-to-one correspondence with the tensor products with unit e on L. This correspondence is given by the formula

$$f \in_a H(X,Y) \text{ iff for every } b \in L,\ x \in_b X \text{ implies } f(x) \in_{a\square b} Y.$$

(In particular, the cartesian closedness - cf.[5] - corresponds to the operation of infimum; in that case, of course, L has to be supposed completely distributive.)

We write

$$f\colon{}_a X \longrightarrow Y \text{ for } f \in_a H(X,Y).$$

The closed category with the closedness structure induced by $\square$ will be denoted by

$$(L, \square)\text{-Fuzz}.$$

§ 2. Dispersed morphisms

2.1. A concrete category $(\mathcal{A}, U)$ is a category $\mathcal{A}$ together with a faithful functor $U: \mathcal{A} \longrightarrow \mathrm{Set}$.

2.2. A dispersion on a concrete category $(\mathcal{A}, U)$ consists of the following data:

(1) a tensored lattice $(L, \square)$,

(2) a concrete category $(\mathcal{B}, V)$, and

(3) functors $F: \mathcal{A} \longrightarrow \mathcal{B}$ and $G: \mathcal{A} \longrightarrow L\text{-Fuzz}$ such

that

(i) $! \circ G \cong U$,

(ii) $? \circ G = V \circ F$, and

(iii) whenever X, Y are objects of $\mathcal{A}$ and $Vf = ?g$ for $f: FX \longrightarrow FY$ and $g: GX \longrightarrow GY$, there is an $h: X \longrightarrow Y$ such that $f = Fh$ and $g = Gh$.

The situation is visualized in the following diagram:

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "U"'] \arrow[d, "G"] \arrow[r, "F"] & \mathcal{B} \arrow[d, "V"] \\
\mathrm{Set} & L\text{-Fuzz} \arrow[l, "!"] \arrow[r, "?"] \arrow[ul, phantom, "\cong"] & \mathrm{Set}
\end{tikzcd}$$

An a-dispersed morphism between objects X, Y of $\mathcal{A}$, written

$$f: X \xrightarrow{\text{a-disp}} Y,$$

is a morphism

$$f: FX \longrightarrow FY$$

of $\mathcal{B}$ such that $Vf:_a GX \longrightarrow GY$ in $(L,\square)$-Fuzz.

2.3. <u>Remarks:</u> 1) The functors G, F are necessarily faithful (we have $! \circ G$ faithful, hence G is; consequently also F, since $V \circ F = ? \circ G$).

2) Consequently, the morphism h in the condition (iii) in 2.2 is uniquely determined by the f. Thus, the functor F establishes a one-to-one correspondence between the morphisms $X \longrightarrow Y$ and the e-dispersed morphisms $X \xrightarrow{e\text{-disp}} Y$.

3) Of the category $\mathcal{B}$, only the full subcategory generated by $F(\mathfrak{A})$ plays a role.

2.4. Let us summarize more intuitively what happens in a dispersion of $(\mathfrak{A},U)$: an object X of $\mathfrak{A}$ carried by UX is represented by an object of $\mathcal{B}$ carried by a fuzzy set M such that $!M$ (i.e. the system of the elements with "full membership" in M) still coincides with UX. The morphisms between thus fuzzily extended objects which are also morphisms in L-Fuzz are unique extensions of the original morphisms (and can be identified with them). At the others, the a in the expression $Vf:_a GX \longrightarrow GY$ represents the degree in which it assimilates a morphism of $\mathfrak{A}$ (the degree of strictness of the values etc.).

2.5. <u>Remark:</u> One sees immediately that

$$f: X \xrightarrow{a\text{-disp}} Y \;\&\; g: Y \xrightarrow{b\text{-disp}} Z \text{ implies } g \circ f: X \xrightarrow{a\square b\text{-disp}} Z.$$

Thus, a dispersion on $(\mathfrak{A},U)$ gives rise to an $(L,\square)$-Fuzz-category $\mathcal{C}$ (see further in 4.7) where $f \in_a \mathcal{C}(X,Y)$ iff $f: X \xrightarrow{a\text{-disp}} Y$ and into which $\mathfrak{A}$ is embedded exactly as its "crisp part". Such $(L,\square)$-Fuzz-categories will be characterized in § 5.

§ 3. Examples

3.1. In [1] an interesting way of representing dispersed morphisms was presented. Roughly speaking, given a monad $\mathbb{T} = (T, \mu, \eta)$ over $\mathfrak{K}$ consider the natural embedding J of $\mathfrak{K}$ into $\mathfrak{K}^{\mathbb{T}}$. It is not full; the morphisms $JX \longrightarrow JY$ which are not in $J(\mathfrak{K})$ represent the newly added generalized morphisms. This construction, already with $\mathfrak{K}$ = Set, covers many of the usual notions of generalized mappings (partial functions, relations, stochastic mappings etc.). We will show now that for $\mathfrak{K}$ = Set the construction from [1] can be viewed as a special case of the dispersion from 2.2. In fact, there holds

Proposition: Let F: Set $\longrightarrow \mathcal{B}$ be a left adjoint to a faithful V: $\mathcal{B} \longrightarrow$ Set. Let L be the lattice consisting of 0 and 1 (there is just one tensor product, namely the infimum, there). Then there is a G: Set $\longrightarrow$ L-Fuzz (unique up to natural equivalence) such that (L, ($\mathcal{B}$,V), F,G) is a dispersion on (Set, 1_{Set}).

Proof: Let $\rho : F \circ V \longrightarrow 1$, $\eta : 1 \longrightarrow V \circ F$ be the adjunction transformations. Since L = {0,1}, the formulas

$$?G = VF, \quad !G(X) = \eta_X(X)$$

uniquely determine a functor G: Set $\longrightarrow$ L-Fuzz. Let f: $FX \longrightarrow FY$, g: $GX \longrightarrow GY$ be such that $Yf = ?g$. Thus, $Vf(\eta_X(X)) \subset \eta_Y(Y)$ and hence there exists an h: $X \longrightarrow Y$ such that

$$Vf \circ \eta_X = \eta_Y \circ h.$$

But we have also $VFh \circ \eta_X = \eta_Y \circ h$, and since $V\varphi \circ \eta$ is the morphism associated with φ in the one-to-one correspondence of the adjunction, $Fh = f$. Since $?Gh = VFh = Vf = ?g$, we have also $Gh = g$.

3.2. Multivalued mappings: Let L be the inversely ordered set of positive natural numbers plus ∞, $\square$ the usual multiplication of numbers. Let $\mathcal{B}$ be the category of all sets of the form $FX = \{A \subset X \mid A \neq \emptyset\}$ and their union preserving mappings, $V: \mathcal{B} \subset$ Set. Define functors F: Set $\longrightarrow \mathcal{B}$, G: Set $\longrightarrow$ L-Fuzz putting FX as above, $Ff(A) = f(A)$, $?(GX) = FX$, $A \in_n G(X)$ iff card $A \leq n$, $?G(f) = F(f)$. Obviously, the condition (iii) is satisfied.

If g: $FX \longrightarrow FY$ in $\mathcal{B}$ and $A \in_m GX$, we have card $g(A) =$ $=$ card $\bigcup \{f(x) \mid x \in A\} \leq \sum_{x \in A}$ card $f(x) \leq m.\sup$ card $f(x)$.

Thus, we see that here g is an n-dispersed mapping $X \longrightarrow Y$ iff it is a multivalued mapping $X \longrightarrow Y$ such that sup card $f(x) \leq$ $\leq n$.

3.3. Stochastic mappings: Let L be the set of non-positive real numbers plus $-\infty$ with the usual order, $\square$ the usual addition. Let I be the unit interval. For a set X define FX as the set

$$\{p: X \longrightarrow I \mid p^{-1}(I \setminus \{0\}) \text{ finite}, \ \sum_X p(x) = 1\}$$

(from now on, we are going to represent the elements of FX as formal linear combinations $\sum_{x \in X} p(x).x$ of elements of X) endowed by the obvious convexity structure (i.e., for $a_i \in I$ such that $\sum_{i=1}^{n} a_i = 1$, $\sum_{i=1}^{n} a_i \sum_X p_i(x).x =$

$= \sum_X \left(\sum_{i=1}^{m} a_i p_i(x)\right).x$).

Define $\mathcal{B}$ as the category the objects of which are the FX, the morphisms are the mappings g for which $g(\sum a_i p_i) =$ $= \sum a_i g(p_i) = \sum a_i g(p_i)$. V: $\mathcal{B} \longrightarrow$ Set is the natural forgetful functor.

Define F: Set $\longrightarrow \mathcal{B}$, G: Set $\longrightarrow$ L-Fuzz as follows:

FX as above, $F(f)(\sum p(x).x) = \sum p(x).f(x)$, ?GX = VFX,

$p \in_a GX$ iff $\sum_X p(x).\log p(x) \geq a$;

obviously, if $p \in_a GX$ implies $f(p) \in_a GY$, f = Fh for a suitable h. Thus, the condition (iii) is satisfied.

Now, let f be an a-dispersed mapping $X \longrightarrow Y$. Thus, we have $f: FX \longrightarrow FY$ in $\mathcal{B}$, hence determined by a formula

$$f(x) = \sum_y f_{xy}.y,$$

and it satisfies, in particular, the inequality

(1) $\inf_{x \in X} \sum_y f_{xy} \log f_{xy} \geq a.$

On the other hand, let (1) hold for an f. We have, for a general $p \in FX$, $f(p)(y) = \sum_x p(x).f_{xy}$ and hence

$$\sum_y f(p)(y).\log f(p)(y) = \sum_y \left(\sum_x p(x).f_{xy}\right).\log\left(\sum_z p(z)f_{zy}\right) =$$
$$= \sum_x p(x) \sum_y f_{xy}.\log\left(\sum_z p(z).f_{zy}\right) \geq \sum_x p(x) \sum_y f_{xy} \log(p(x).f_{xy}) =$$
$$= \sum_x p(x).\log p(x). \sum_y f_{xy} + \sum_x p(x). \sum_y f_{xy}.\log f_{xy} \geq$$
$$\geq \sum p(x).\log p(x) + a,$$

so that f is an a-dispersed mapping.

Thus, here f is an a-dispersed mapping iff it is a stochastic mapping with the "informational dispersion"

$\sup \left(- \sum f_{xy}.\log f_{xy}\right) \leq |a|$.

3.4. Dispersed contractions: Let L be the inversely ordered set of non-negative real numbers, $\square$ the addition. Let $(\mathcal{A},U)$ be the category of metric spaces and contractions. For a metric space define FX as its Hausdorff superspace (see e.g.[3]; FX is the set of all non-void compact subsets of X endowed by the metric

$$\rho^*(A,B) = \max(\max_{x\in A} \rho(x,B), \max_{x\in B} \rho(y,A)).)$$

Let $\mathcal{B}$ be the category of all the spaces of the form FX and their contractions such that $f(A) = \bigcup\{f(x) \mid x\in A\}$, $V: \mathcal{B} \longrightarrow$ Set the natural forgetful functor. Define Ff for $f: X\longrightarrow Y$ by $Ff(A) = f(A)$. $G: \mathcal{A} \longrightarrow$ L-Fuzz is defined by $?G = VF$ with $A \in_a GF$ iff diam $A \leq a$ (since diam $f(A) \leq$ diam A, this definition is correct). A mapping $g: FX \longrightarrow FY$ is an a-dispersed mapping $X \longrightarrow Y$ iff diam $g(\{x\}) \leq a$ for every $x \in X$. (If sup diam $g(\{x\}) \leq a$ we have diam $g(A) \leq$ diam $A + a$. Really, consider $x_i \in A$, $u_i \in g(x_i)$, $i = 1,2$; since g is a contraction with respect to ρ^* above, we have $d =$ diam $A \geq \rho^*(g(x_1), g(x_2))$, hence $\rho(u_1,z) \leq d$ for a $z \in g(x_2)$ and hence $\rho(u_1,u_2) \leq \rho(u_1,z) + \rho(z,u_2) \leq d + a$.)

3.5. Remark: In all the examples, there was a generator I of $\mathcal{A}$ (the one-point set or space) and a natural equivalence $\varkappa : \mathcal{A}(I,-) \cong {!G}$ such that for every $x \in_a GX$ one had a (unique) $\xi : FI \longrightarrow FX$ with $V(\xi)(\varkappa(1_I)) = x$ and $V(\xi) :_a GI \longrightarrow GX$. This property will play a role in the sequel.

§ 4. Praedispersions and fuzzy extensions

4.1. Convention: Throughout this and the following paragraph we will use the symbol

$$I$$

for a fixed generator of the category in question. Thus, if there is no danger of confusion, we write $F(I) = I$ in the case of a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ just to indicate that FI is again a generator of $\mathcal{B}$ (not necessarily really identical with the $I \in \operatorname{obj} \mathcal{A}$).

A concrete category $(\mathcal{A}, U)$ in which the forgetful functor is naturally equivalent to $\mathcal{A}(I,-)$ will be indicated by

$$(\mathcal{A}, I).$$

4.2. An $(L, \Box)$-<u>praedispersion</u> ($(L, \Box)$ is a tensored lattice) $\mathcal{D} = (\mathcal{B}, V, G, G)$ over a concrete category $(\mathcal{A}, I)$ consists of

a concrete category $(\mathcal{B}, V)$,

a one-to-one functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, and

a functor $G\colon \mathcal{A} \longrightarrow$ L-Fuzz

such that

$F(\operatorname{obj} \mathcal{A}) = \operatorname{obj} \mathcal{B}$, and

$V \circ F = ? \circ G.$

The following special conditions on praedispersions will be considered:

(a): There is a natural equivalence

$$\varkappa : \mathcal{A}(I,-) \cong \,!G.$$

(a*): (a) & , moreover,

for every $x \in_a GX$ there is exactly one $\xi : FI \longrightarrow FX$ such that $V(\xi)(\varkappa(1_I)) = x$ and $V(\xi) :_a GI \longrightarrow GX$.

(b): For any two morphisms $f\colon FX \longrightarrow FY$, $g\colon GX \longrightarrow GY$ such that $Vf = ?g$ there is an $h\colon X \longrightarrow Y$ such that $f = Fh$ and

$g = Gh$.

4.3. Remarks: 1) Since F is one-to-one, G is faithful, and if (b) holds, there is exactly one required h.

2) For a concrete category $(\mathcal{A}, I)$, the dispersion from 2.2 is a praedispersion satisfying (a) and (b). Moreover, all the examples from § 3 satisfy (a*) (see 3.5).

4.4. Let $\mathcal{D}_i = (\mathcal{B}_i, V_i, G_i)$ be praedispersions over $(\mathcal{A}_i, I)$ $(i = 1,2)$.

We say that $\mathcal{D}_1$ is equivalent to $\mathcal{D}_2$ and write

$$\mathcal{D}_1 \sim \mathcal{D}_2$$

if there are isofunctors

$$E : \mathcal{B}_1 \cong \mathcal{B}_2 \quad , \tilde{E} : \mathcal{A}_1 \cong \mathcal{A}_2$$

and natural equivalences

$$\varepsilon : V_1 \cong V_2 E, \quad \tilde{\varepsilon} : G_1 \cong G_2 E$$

such that $\tilde{E}(I) = I$, $E \circ F_1 = F_2 \circ \tilde{E}$ and $?\tilde{\varepsilon} = \varepsilon F_1$.

4.5. Remarks: 1) Obviously, $\sim$ is reflexive, symmetric and transitive.

2) One sees easily that $\mathcal{D}_1 \sim \mathcal{D}_2$ iff there is an isofunctor $E : \mathcal{B}_1 \cong \mathcal{B}_2$ and a natural equivalence $\varepsilon : V_1 \cong$ $\cong V_2 \circ E$ such that $EF_1(I) = F_2(I)$, $E(F_1(\mathcal{A}_1) = F_2(\mathcal{A}_2)$ and

$$\text{for } x \in_a G_1(X) \qquad \varepsilon(x) \in_a G_2 F_2^{-1} EF_1(X)$$

(and that in such a case the $\tilde{E}$ and $\tilde{\varepsilon}$ are uniquely determined).

4.6. Proposition: Let $\mathcal{D}_1 \sim \mathcal{D}_2$. If $\mathcal{D}_1$ satisfies (a), (a*), (b), respectively, so does $\mathcal{D}_2$.

__Proof:__ We will just give the formulas, omitting the tedious checking.

(a) $\varkappa_2 \colon \mathcal{A}_2(I,-) = {!G_2}$ is obtained as

$$\mathcal{A}_2(I,-) = \mathcal{A}_2(I,\tilde{E}-) \circ \tilde{E}^{-1} \xrightarrow{\tau\tilde{E}^{-1}} \mathcal{A}_1(I,-),$$

$$\tilde{E}^{-1} \xrightarrow{\varkappa\tilde{E}^{-1}} {!G_1\tilde{E}^{-1}} \xrightarrow{!\,\tilde{\varepsilon}\tilde{E}^{-1}} {!G_2}$$

where $\tau(\alpha) = \tilde{E}^{-1}(\alpha)$.

(a$*$) For $x \in_a G_2(X)$ we have $y = \tilde{\varepsilon}^{-1}E^{-1}(x) \in_a G_1E^{-1}(X)$ for which there is an $\eta : F_1I \longrightarrow F_1\tilde{E}^{-1}(X)$ such that $V_1(\eta)(\varkappa_1(1)) = y$ and $V_1(\eta) : {}_aG_1I \longrightarrow G_1E^{-1}(X)$. Consider $\xi = E\eta : F_2I = EF_1I \longrightarrow EF_1\tilde{E}^{-1}X = F_2X$.

(b) Let $V_2f = ?g$ for $f\colon F_2X \longrightarrow F_2Y$, $g\colon G_2X \longrightarrow G_2Y$. We have $?(\varepsilon^{-1} \circ g \circ \varepsilon) = V_1(E^{-1}f)$, hence there is an $\bar{h}$ such that $E^{-1}f = F_1\bar{h}$ and $\varepsilon^{-1} \circ g \circ \varepsilon = G_1\bar{h}$. Put $h = E\bar{h}$.

4.7. For a notion of a $\mathcal{V}$-category where $\mathcal{V}$ is a closed category see e.g. [4]. In particular, an $(L,\square)$-Fuzz-category $\mathcal{C}$ consists of a class obj $\mathcal{C}$ of objects, L-fuzzy sets $\mathcal{C}(X,Y)$ associated with couples of objects, associative composition

$$\circ : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$$

(i.e., an associative composition

$$\circ : ?\,\mathcal{C}(Y,Z) \times ?\,\mathcal{C}(X,Y) \longrightarrow ?\,\mathcal{C}(X,Z)$$

such that for $\beta \in_b \mathcal{C}(Y,Z)$ and $\alpha \in_a \mathcal{C}(X,Y)$, $\beta \circ \alpha \in_{b \square a} (X,Z))$, and units $1_X \in_e \mathcal{C}(X,X)$ such that for $\alpha \in$ $\in ?\,\mathcal{C}(X,Y)$ $\alpha \circ 1_X = 1_Y \circ \alpha = \alpha$.

For an $(L, \square)$-Fuzz-category $\mathcal{C}$ define categories

$$?\mathcal{C}\ ,\ !\mathcal{C}$$

Putting

$$\mathrm{obj}\ ?\mathcal{C} = \mathrm{obj}\ !\mathcal{C} = \mathrm{obj}\ \mathcal{C}\ ,$$

$$(?\mathcal{C})(X,Y) = ?(\mathcal{C}(X,Y)),\ (!\mathcal{C})(X,Y) = !\ (\mathcal{C}(X,Y)).$$

4.8. An $(L, \square)$-extension of a concrete category $(\mathcal{A}, I)$ is an $(L, \square)$-Fuzz-category $\mathcal{C}$ such that there is an isofunctor $H: \mathcal{A} \cong !\mathcal{C}$ such that FI is a generator of both $!\mathcal{C}$ and $?\mathcal{C}$.

The following special conditions on $(L, \square)$-Fuzz-categories $\mathcal{C}$ with a common generator I of $!\mathcal{C}$ and $?\mathcal{C}$ will be considered:

(c) If $f: X \longrightarrow Y$ in $?\mathcal{C}$ is such that

$$\alpha \in_a \mathcal{C}(I,X) \text{ implies } f \circ \alpha \in_a \mathcal{C}(I,Y),$$

then $f \in_e \mathcal{C}(X,Y)$.

(c*) If $f: X \longrightarrow Y$ in $?\mathcal{C}$ is such that

$$\alpha \in_b \mathcal{C}(I,X) \text{ implies } f \circ \alpha \in_{a \square b} \mathcal{C}(I,Y),$$

then $f \in_a \mathcal{C}(X,Y)$.

4.9. $(L, \square)$-Fuzz-categories $\mathcal{C}_i$ $(i = 1,2)$ are said to be isomorphic (we write $\mathcal{C}_1 \sim \mathcal{C}_2$) if there is an isofunctor $E: ?\mathcal{C}_1 \cong ?\mathcal{C}_2$ such that

$$f \in_a \mathcal{C}_1(X,Y) \text{ iff } Ef \in_a \mathcal{C}_2(X,Y).$$

4.10. Proposition: Let $\mathcal{C}_1 \sim \mathcal{C}_2$. If $\mathcal{C}_1$ satisfies (c), (c*), respectively, so does $\mathcal{C}_2$.

Proof is trivial.

§ 5. Extensions respresenting dispersions

We observed in 2.5 that, (in the terminology of 4.8) a dispersi om over a category gives rise to its extension. We will show now that, roughly speaking, the dispersions satisfying (a$*$) may be characterized as the extensions satisfying (c$*$).

5.1. (cf. 2.5.) Let $\mathcal{D} = (\mathcal{B}, V, F, G)$ be a praedispersion over $(\mathcal{A}, I)$. We associate with $\mathcal{D}$ an $(L, \square)$-Fuzz-category $\mathcal{C}$ as follows:

$\mathrm{obj}\,\mathcal{C} = \mathrm{obj}\,\mathcal{A}$,

$f \in_a \mathcal{C}(X,Y)$ iff $f\colon FX \longrightarrow FY$ and $Vf\colon {}_aGX \longrightarrow GY$

(composition as in $\mathcal{B}$).

The situation will be indicated by

$$\mathcal{D} \longmapsto \mathcal{C} .$$

5.2. Proposition: If $\mathcal{D}_1 \sim \mathcal{D}_2$ and $\mathcal{D}_i \longmapsto \mathcal{C}_i$ then $\mathcal{C}_1 \sim \mathcal{C}_2$.

Proof: Consider the isofunctor $E\colon \mathcal{B}_1 = ?\mathcal{C}_1 \longrightarrow \mathcal{B}_2 = ?\mathcal{C}_2$ and the natural equivalence $\varepsilon : V_1 \longrightarrow V_2E$. We have

$$V_2(Ef) = \varepsilon \circ V_1 f \circ \varepsilon^{-1}.$$

Let $f \in_a \mathcal{C}_1(X,Y)$. Hence, $V_1 f\colon_a G_1X \longrightarrow G_1Y$. For an $x \in_b G_2(EX)$ we have (see 4.5.2) $\varepsilon^{-1}(x) \in_b G_1X$, hence $V_1f(\varepsilon^{-1}(x)) \in_{a \square b} \in_{a \square b} G_1Y$ and hence $V_2(Ef)(x) \in_{a \square b} G_2EY$. Thus, $Ef \in_a \mathcal{C}_2(EX,EY)$. Using the fact that $V_1 f = \varepsilon^{-1} \circ V_2(Ef) \circ \varepsilon$ we see analogously the converse.

5.3. With an $(L, \square)$-Fuzz-category $\mathcal{C}$ having a common generator I for $!\mathcal{C}$ and $?\mathcal{C}$ associate the praedispersion $\mathcal{D} = (?\mathcal{C}, ?\mathcal{C}(I,-), !\mathcal{C} \subset ?\mathcal{C}, \mathcal{C}(I,-))$. The situation will be indicated by

$$\mathcal{C} \longmapsto \mathcal{D} .$$

5.4. <u>Proposition:</u> If $\mathcal{C}_1 \sim \mathcal{C}_2$ and $\mathcal{C}_i \longmapsto \mathcal{D}_i$, then $\mathcal{D}_1 \sim \mathcal{D}_2$.

<u>Proof:</u> We have E: $?\mathcal{C}_1 \cong ?\mathcal{C}_2$ with the property from 4.9. In particular, $E(!\mathcal{C}_1) = !\mathcal{C}_2$. Define

$$\varepsilon : ?\mathcal{C}_1(I,-) \longrightarrow ?\mathcal{C}_2(I,E-)$$

putting $\varepsilon(\alpha) = E(\alpha)$. This is a natural equivalence and we have $\varepsilon(\alpha) \in_a \mathcal{C}_2(I,EX)$ for $\alpha \in_a \mathcal{C}_1(I,X)$. Thus, the statement follows by 4.5.2.

5.5. <u>Proposition:</u> Let $\mathcal{C} \longmapsto \mathcal{D}$. Then $\mathcal{D}$ satisfies (a$*$).

<u>Proof:</u> Take $\varkappa$ = ident: $!\mathcal{C}(I,-) \cong !\mathcal{C}(i,-)$. Then $V(\xi)(\varkappa(I)) = ?\mathcal{C}(I,\xi)(1) = \xi$ which yields immediately the uniqueness. If $\xi \in_a \mathcal{C}(I,X)$ and if $\alpha \in_b \mathcal{C}(I,I)$, we have $V(\xi)(\alpha) = \xi \circ \alpha \in_{a \square b} \mathcal{C}(I,X)$ so that $V(\xi):_a \mathcal{C}(I,I) \longrightarrow \mathcal{C}(I,X)$.

5.6. <u>Proposition:</u> Let $\mathcal{C}$ satisfy (c), let $\mathcal{C} \longmapsto \mathcal{D}$. Then $\mathcal{D}$ satisfies (b).

<u>Proof:</u> Let $f: X \longrightarrow Y$ in $?\mathcal{C}$ and $g: \mathcal{C}(I,X) \longrightarrow \mathcal{C}(I,Y)$ be such that $?\mathcal{C}(I,f) = ?g$. Then $g = \mathcal{C}(I,f)$. Hence, if $\alpha \in_a \mathcal{C}(I,I)$, we have $f \circ \alpha = g(\alpha) \in_a \mathcal{C}(I,X)$, so that, by (c), $f: X \longrightarrow Y$ in $!\mathcal{C}$.

5.7. <u>Proposition:</u> Let $\mathcal{D}$ satisfy (a$*$), let $\mathcal{D} \longmapsto \mathcal{C}$. Then $\mathcal{C}$ satisfies (c$*$).

<u>Proof:</u> Let $f: FX \longrightarrow FY$ in $\mathcal{B} = ?\mathcal{C}$ be such that

$$\alpha \in_b \mathcal{C}(I,X) \text{ implies } f \circ \alpha \in_{a \square b} \mathcal{C}(I,Y).$$

Thus, if $\alpha : FI \longrightarrow FX$ is such that $V\alpha :_b GI \longrightarrow GX$, we have $V(f\alpha) = Vf \circ V\alpha :_{a \sqcap b} GI \longrightarrow GY$. For an $x \in_b GX$ take the $\xi : FI \longrightarrow FX$ such that $V(\xi)(\varkappa(1)) = x$ and $V(\xi) :_a GX \longrightarrow$ $\longrightarrow GY$. Thus,

$$V(f)(x) = V(f \circ \xi)(\varkappa(1)) \in_{a \sqcap b} GY$$

so that $Vf :_a GX \longrightarrow GY$ and hence $f \in_a \mathcal{C}(X,Y)$.

5.8. Proposition: Let $\mathcal{C}$ satisfy $(c*)$, let

$$\mathcal{C} \longmapsto \mathcal{D} \longmapsto \mathcal{C}' .$$

Then

$$\mathcal{C} \sim \mathcal{C}' .$$

Proof: We have $?\mathcal{C}' = ?\mathcal{C}$. Put $E = 1_{?\mathcal{C}}$. We have $f \in_a \mathcal{C}(X,Y)$, iff $\alpha \in_b \mathcal{C}(I,X)$ implies $f \circ \alpha \in_{a \sqcap b} \mathcal{C}(I,Y)$, i.e. iff $\alpha \in_b \mathcal{C}(I,X)$ implies $?\mathcal{C}(I,f)(\alpha) \in_{a \sqcap b} \mathcal{C}(I,Y)$. thus, iff $f \in_a \mathcal{C}'(X,Y)$.

5.9. Proposition: Let $\mathcal{D}$ satisfy $(a*)$ and (b), let

$$\mathcal{D} \longmapsto \mathcal{C} \longmapsto \mathcal{D}' .$$

Then

$$\mathcal{D} \sim \mathcal{D}' .$$

Proof: Put $\mathcal{D} = (\mathcal{B}, V, F, G)$. Define $E : \mathcal{C} \cong \mathcal{B}$ putting $EX = FX$ for objects, $Ef = f$ for morphisms,

$$\varepsilon : ?\mathcal{C}(U,-) \longrightarrow V \circ E$$

putting

$$\varepsilon_X(\xi) = V(\xi)(\varkappa(1)).$$

One checks easily that it is a natural transformation. By $(a*)$, every ε_X is invertible so that ε is a natural equivalence. We have to prove that

I. $!\mathcal{C} = E(!\mathcal{C}) = F(\mathcal{A})$, and that

II. $\xi \in_a \mathcal{C}(I,X)$ implies $\varepsilon(\xi) \in_a GX$.

I: If $f\colon X \longrightarrow Y$ in $!\mathcal{C}$ we have $Vf\colon_e GX \longrightarrow GY$, hence $Vf = ?g$ for a $g\colon GX \longrightarrow GY$. Thus, by (b), there is an h with $f = Fh$. On the other hand, for $f = Fh$, $h\colon X \longrightarrow Y$, we have $Vf = VFh = ?Gh\colon_e GX \longrightarrow GY$, so that $f \in !\mathcal{C}$.

II: If $\xi \in_a \mathcal{C}(I,X)$, we have $\xi\colon FI \longrightarrow FX$ in $\mathcal{B}$, $V(\xi)\colon_a GI \longrightarrow GX$. Thus, $\varepsilon(\xi) = V(\xi)(\varkappa(1)) \in_a GX$.

5.10. Let us summarize the statements of 5.5 - 5.9. See the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[d, mapsto] & \\
\mathcal{D}(a*) \arrow[d, mapsto] & \mathcal{C}(c) \arrow[d, mapsto] \\
\mathcal{C}(c*) \arrow[r, shift left] & \mathcal{D}(a*,b) \arrow[l, shift left]
\end{tikzcd}$$

Starting with a general $\mathcal{C}$ one goes over to a $\mathcal{D}$ satisfying (a$*$), from this we obtain a $\mathcal{C}$ satisfying (c$*$). Such $\mathcal{C}$ are already in a one-to-one correspondence wtih the dispersions satisfying (a$*$). Thus, an $(L,\square)$-Fuzz-category represents a dispersion (satisfying (a$*$)) of its crisp part iff it satisfies ($*$).

5.11. To illustrate what happens let us compare two extensions of the category of metric spaces and contractions. The first one was described in 3.4, for the second one let us take the Lipschitz mappings (L is the inversely ordered set of real numbers ≥ 1, $\square$ is the usual multiplication, $f \in_a \mathcal{C}(X,Y)$ iff $\varphi(f(x),f(y)) \leq a.\ \varphi(x,y)$). Unlike in the first case, in the second one if we start with the given $\mathcal{C}$, proceed to $\mathcal{D}$ and back to $\mathcal{C}'$ we have

$$!\mathcal{C}' = ?\mathcal{C}' = ?\mathcal{C} \quad .$$

R e f e r e n c e s :

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