COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,2 (1976)

METRIC-FINE, PROXIMALLY FINE, AND LOCALLY FINE UNIFORM SPACES

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Abstract: The following main result is established in the paper. A metric-fine (measurable) proximally fine space is locally fine if and only if the space is proximally fine and each uniformly locally finite cozero (Baire) cover is a uniform cover if and only if each hypercozero(hyperBaire) set is a cozero (Baire) set.

Key Words and Phrases: Metric-fine, measurable proximally fine, cozero-fine, Baire-fine, locally fine uniform spaces; uniformly locally finite uniform cover; cozero set, Baire set, hypercozero set, hyperBaire set.

AMS: Primary 54E15, 54 C50 Ref. Ž.: 3.962
Secondary 54H05

This paper originated in the attempt to establish that metric-fine proximally fine spaces were locally fine. This question has since been answered in the negative by the author ($[R]_3$). (A negative answer to this question is also implicit in $[Fr]_3$, in view of the correction by [P].) The main contributions of the present work are Theorem 2.1, which shows that the condition hypercozero=cozero guarantees the locally fine property for metric-fine proximally fine spaces and Theorem 2.2.

In general, the notation employed is found in $[R]_{1-5}$ and [I], and is consistent with the terms used in $[Fr]_{1-8}$. uX denotes a uniform space. If u and v are uniformities, u/v is the quasi-uniformity having covers of the form $\{V_s \cap U_t^s\}$ as a basis, where $\{V_s\} \in v$, $\{U_t^s\} \in u$, for each s. uX is locally fine if $u/u + u^{(1)} = u$ and locally sub-M-fine if $u(eu = u$, where eu has the basis of countable u-covers. A function $uX \xrightarrow{f} vY$ is ULUC if f/U_s is uniformly continuous for each member of $\{U_s\} \in u$.

Theorem 1.1: These statements are equivalent.

(i) uX is metric-fine and each bounded ULUC function is uniformly continuous. (uX is locally e-fine metric-fine in the sense of $[Fr]_2$.)

(ii) uX is metric-fine and hypercoz (uX) = coz (uX).

(iii) Each σ-uniformly discrete cozero cover is a uniform cover.

(iv) uX is locally sub-M-fine and each uniformly locally countable cozero cover is a uniform cover.

Proof: The equivalence of (i) - (iii) has been established in $[R]_5$ and $[Fr]_2$, Theorem 3, while (iv) follows from (i) using $[Fr]_2$, Theorem 3, and the definition of locally e-fine. We sketch a proof of (i) $\longrightarrow$ (iv) that also enables us to establish 2.2. Let $\{\mathrm{coz}\, f_t\}$ be a uniformly locally countable cozero cover with respect to $\mathcal{S}_p(\varepsilon)$, where φ is uniformly continuous. Let $\mathcal{U} = \bigcup \mathcal{U}_i$, $\mathcal{U}_i = \{U_{s,i} : s \in S_i\}$, be a σ-uniformly discrete uniform refinement of $\mathcal{S}_\varphi(\varepsilon/4)$, with $\mathcal{U}_i$ discrete with respect to $\mathcal{S}_{\varphi_i}(\varepsilon_i)$, φ_i uniformly continuous, $\varepsilon_i < \varepsilon$. For

$s \in S_i$, define the cozero sets $V_{s,i} = \{x: \varphi_i(x, U_{s,i}) < \varepsilon_i/8\}$ and the countable family $C_{s,i} = \{\text{coz } f_t: \text{coz } f_t \cap V_{s,i} \neq \emptyset\}$. Write $C_{s,i} = \{S^j_{s,i}: j \in N\}$ and for $j \in N$ define $T^j_{s,i} = S^j_{s,i} \cap V_{s,i}$; then the cozero family $\{T^j_{s,i}: s \in S_i\}$ is uniformly discrete for each j, so by (ii) $C^j_i = \bigcup \{T^j_{s,i}: s \in S_i\}$ is a cozero set. Define $B_i = \{x: \rho(x, U_{s,i}) > \varepsilon_i/16$ for all $s \in S_i\}$ and let $\mathcal{V}_i = \{C^j_i: j \in N\} \cup \{B_i\}$. By (iii), $\mathcal{V}_i \in u$. Define $H_i = \bigcup \{U_{s,i}: s \in S_i\}$ and set $\mathcal{A}_i = \mathcal{V}_{i/H_i}$. Note that $\mathcal{A}_i$ is a uniform cover of H_i. Finally, $\mathcal{U}_i \wedge \mathcal{A}_i < \{\text{coz } f_t\}_{/H_i}$; hence $\{\text{coz } f_t\} \in u/eu = u$ since uX is metric-fine.

Assume that (iv) is satisfied. Then each countable cozero cover is uniform and uX is locally sub-M-fine, so uX is metric-fine ([R]$_5$). Let $X \xrightarrow{f} [0,1]$ be a UIUC function with respect to $\mathcal{U} = \{U_s\}$, where $\mathcal{U}$ is a uniformly locally finite cozero cover (which may be assumed since uX is metric-fine). If $\{H_i\}$ is a finite open cover of $[0,1]$, then $\{U_s \cap \cap f^{-1}(H_i)\}$ is a uniformly locally finite cozero cover that refines $\{f^{-1}(H_i)\}$; hence by (iv) f is uniformly continuous and (i) is established.

Theorem 1.2: Assume that uX has a basis of finite dimensional uniform covers. Then each countable (resp. finite) cozero cover is a uniform cover and hypercoz (uX) = coz (uX) if and only if each uniformly locally countable (resp. uniformly locally bounded) cozero cover is a uniform cover.

Proof: Using the notation in 1.1 and the fact ([I], 4.25) that each uniform cover has a uniform refinement which is the finite union of uniformly discrete families, we may assume $\mathcal{U} = \bigcup \mathcal{U}_i$, where i ranges over a finite set. The proof of 1.1 now proceeds unaltered to the conclusion that $\{\operatorname{coz} f_t\}$ is a uniform cover, since it is uniform on each member of the finite uniform cover $\{H_i\}$.

Note (i): The uniformly locally bounded assumption in 1.2 cannot be replaced by uniformly locally finite. The referee points out that if ρ is the usual metric on R and is the fine uniformity on R, then $\rho \vee p\alpha$ satisfies the conditions in 1.2 for uniformly locally finite, but each such cover is not uniform (since $\alpha \neq \rho \vee p\alpha$).

Note (ii): Theorems 1.1 and 1.2 remain valid (using the preceding proofs) if one replaces coz (uX) by Baire (uX) and metric-fine by measurable.

Theorem 2.1: These statements are equivalent.

(i) uX is cozero-fine and locally fine

(ii) uX is cozero-fine and hyper coz (uX) = coz (uX)

(iii) uX is proximally fine and each uniformly locally finite cozero cover is a uniform cover.

Proof: Using 1.1 and the fact that cozero-fine is equivalent to metric-fine and proximally fine ($[H]_3$, 5.3 or $[Fr]_6$, Theorem 5), one easily establishes the implications (i) $\longrightarrow$ (ii) $\longrightarrow$ (iii). Assume that (iii) is satisfied. Let $uX \xrightarrow{f} M$ be a cozero function to the metric space M. Since uX is proximally fine, f is uniformly continuous once it is shown that $f^{-1}\{H_i\} \in u$ for each finite open cover $\{H_i\}$ of M. But each $H_i \in \operatorname{coz}(M)$, so $f^{-1}\{H_i\}$ is a finite cozero cover; hence $f^{-1}\{H_i\} \in u$ by (iii). Thus uX is cozero-fine and has a basis of point-finite uniform covers.

To show that uX is locally fine, it suffices to show that $p(u^{(1)}) = pu$, for uX is proximally fine and $u^{(1)}$ is a uniformity since uX has a point-finite basis. Choose $\mathcal{U} \in p(u^{(1)})$. There exists $\mathcal{V} = \{V_s \cap U_t^s\} \in u^{(1)}$ and a finite cover $\{H_i\}$ such that $\mathcal{V} < \{H_i\} < \mathcal{U}$. Define $S_{s,i} = \cup\{U_t^s : V_s \cap U_t^s \subset H_i\}$ and set $\mathcal{S}_s = \{S_{s,i}\}$. Each $\mathcal{S}_s$ is a finite uniform cover (since $\{U_t^s\} < \mathcal{S}_s$); hence $\mathcal{W} = \{V_s \cap S_{s,i}\} \in pu/u$ and $\mathcal{W} < \mathcal{U}$. Since uX is metric-fine we may assume that $\{V_s\}$ is a uniformly locally finite cozero cover, so by (iii) $\mathcal{W}$, and hence $\mathcal{U}$, is a uniform cover and $p(u^{(1)}) = pu$.

<u>Theorem 2.2</u>: These statements are equivalent.

(i) uX is Baire-fine and locally fine.

(ii) uX is Baire-fine and hyperBaire (uX) = Baire (uX).

(iii) uX is proximally fine and each uniformly locally finite Baire cover is a uniform cover.

(iv) uX is proximally fine and each σ-uniformly locally finite Baire cover is a uniform cover.

<u>Proof</u>: The equivalence of (i) - (iii) may be established using the comments following 1.2 and the proof technique of 2.1. To establish (i) $\longrightarrow$ (iv), let $\mathcal{U} = \cup \mathcal{U}_i$ be a Baire (= cozero) cover, where $\mathcal{U}_i$ is uniformly locally finite with respect to $\mathcal{V}_i \in u$. Define $B_i = \cup\{U \in \mathcal{U}_i\}$. Then one easily shows that B_i is a cozero set since uX is locally fine (if $U = coz(f_U)$, then $B_i = coz(f)$, where $f = \Sigma f_U$). Also $\mathcal{U}_{i/B_i}$ is a uniform cover of B_i (for its restriction to

each member of $\mathcal{V}_i$ has a finite Baire refinement and uX is measurable and locally fine). Hence uX measurable implies $\mathcal{U} = \{B_i \cap U: U \in \mathcal{U}_i\} \in u/eu = u$.

The reader should compare (i) and (ii) of 2.2 with Theorem 3 of $[Fr]_7$. It has been mentioned that there exist Baire-fine spaces that are not locally fine ($([R]_{2,3})$). In fact, the smallest measurable uniformity u satisfying hyper-Baire (uX) = Baire (uX) which contains the product uniformity of $X = D^{\omega_1}$, where D is uniformly discrete and $|D| = \omega_1$, is not locally fine ($[Fr]_2$, p. 246). On the other hand, ($[R]_2$, 2.6) establishes that if the smallest measurable uniformity u containing a metric uniformity satisfies hyper Baire (uX) = Baire (uX), then uX is locally fine.

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(Oblatum 26.7. 1974)