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FREE UNIFORM MEASURES ON SUB-INVERSION-CLOSED SPACES

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Abstract: Any free uniform measure on any sub-inversion-closed uniform space is represented by a Radon measure with a compact support in the completion of the space.

Relation of free uniform, σ-additive and order-bounded measures is discussed.

Key Words: Free uniform measures, order-bounded and σ-additive functionals, sub-inversion-closed uniform spaces, separable Riesz measures, Riesz measures.

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§ 1. Introduction. The notion "free uniform measure" on a uniform space [1],[3],[15] provides a common generalization for both the notions "Riesz measure" and "separable Riesz measure" (see § 7 below).

It is the aim of this paper to show that the theorem about representation of these measures by means of certain Radon measures - proved by Hewitt ([11], Th. 17) for Riesz measures and by Haydon [10] for separable Riesz measures - holds for free uniform measures on any sub-inversion-closed uniform space (Theorem 4.3 below).

In §§ 5,6 I discuss the connections of free uniform measures with order-bounded and σ-additive functionals on the space of uniform functions.

Terminology and notation. Basic topics on uniform spaces may be found in the Isbell's book [12] but here we shall work rather with pseudometrics than with coverings. All topologies and uniformities are assumed to be Hausdorff.

For a compact topological space C, a Radon measure on C is a (signed) regular Borel measure on C. All Radon measures on C are in one-to-one correspondence with all norm-continuous linear functionals on the Banach space of real-valued functions on C ([17], II - § 2, Ex. 3).

In the whole paper R denotes the reals; X denotes an arbitrary (Hausdorff) uniform space. $\hat{X}$ is the completion of X. $\mathcal{P}(X)$ is the system of all bounded uniformly continuous pseudometrics on X. U(X) is the linear lattice of all uniform (= uniformly continuous) real-valued functions on X, endowed with the topology of pointwise convergence on X.

A set $S \subset U(X)$ is called U.E.-set iff it is equiuniform (= uniformly equicontinuous) and pointwise bounded. A linear form μ on the space U(X) is called free uniform measure iff it is continuous on each U.E.- set in the topology of pointwise convergence. The reader is referred to [15] for basic properties of the space $\mathcal{M}_F(X)$ of free uniform measures on X. Here I shall only add that a set $S \subset U(X)$ is U.E. if and only if its unique extension $\hat{S}$ to $\hat{X}$ is a U.E.-set. Hence the space $\mathcal{M}_F(X)$ and $\mathcal{M}_F(\hat{X})$ are canonically isomorphic.

The Banach space of bounded uniform functions on X will be denoted $U_b(X)$ (the norm is given by $\|f\| = \sup \{ |f(x)| \mid x \in X \}$). Continuous linear forms on the spaces $U_b(X)$ are called measures on X . Here I shall call "measure on X" also a linear form on the space $U(X)$ whose restriction to $U_b(X)$ is measure. Thus $\mu : U(X) \longrightarrow R$ is a measure iff μ is linear and $\|\mu\| = \sup \{ |\mu(f)| \mid f \in U_b(X) \ \& \ \|f\| \leq 1 \}$ is finite. It is easy to see that each free uniform measure is actually a measure.

If $\{f_\alpha\}_{\alpha \in A}$ is a net of real-valued functions on X indexed by elements of a directed set A then the symbol $f_\alpha \searrow 0$ means that $\lim_{\alpha \in A} f_\alpha = 0$ pointwise (i.e. $\lim f_\alpha(x) = 0$ for any $x \in X$) and $f_\alpha \geq f_\beta$ for $\alpha \leq \beta$.

§ 2. Sub-inversion-closed uniform spaces. A subset C of uniform space X is a Coz-set iff there exists a function $f \in U(X)$ such that $C = \{ x \in X \mid f(x) > 0 \}$. A real-valued function g on X is a Coz-function iff the preimage of any open subset of R under g is a Coz-set in X.

A space X is called inversion-closed iff every real-valued Coz-function on X is uniform. The following theorem will not be used below; it is included here just for the reader's orientation. The condition (b) explains the name "inversion-closed" while the condition (c) suggests that this class of uniform spaces should be important in the theory of σ-additive measures.

Theorem. For a uniform space X the three conditions are equivalent:

(a) X is inversion-closed;

(b) if $f \in U(X)$ and $f(x) \neq 0$ for each $x \in X$ then $\frac{1}{f} \in U(X)$;

(c) if $f_n \in U_b(X)$ for $n = 1,2,\dots$ and $f_n \searrow 0$ then the set $\{f_n \mid n = 1,2,\dots\}$ is equiuniform.

Proof will not be repeated here. Implication (a) $\Longrightarrow$ (c) was proved by Preiss and Zahradník [19]. The other implications are proved in Frolík's papers [6],[7] where also other characterizations of inversion-closed spaces are given.

The following property will be used below: any uniform real-valued function on a subspace of an inversion-closed space can be extended to a uniform function on the whole space [8] (this follows from the fact that a Coz-function defined on complement of a Coz-set can be extended to a Coz-function on the whole space).

A uniform space will be called sub-inversion-closed iff it is uniformly isomorphic with a subspace of an inversion-closed space (this class of spaces was pointed out to me by Zdeněk Frolík).

Every inversion-closed space is sub-inversion-closed. Clearly every precompact space is sub-inversion-closed. Moreover, it can be deduced from ([12], VII.9) that every locally fine space is sub-inversion-closed.

§ 3. Supports of uniform measures. Although we shall work only with free uniform measures all results in this paragraph hold for all uniform measures (with the same proofs).

3.1. Notation. If $\varphi \in \mathcal{P}(X)$ put $\varphi^x(y) = (1 - \varphi(x,y))^+$ for $x,y \in X$; obviously $\varphi^x \in U_b(X)$, $\varphi^x \geq 0$. For any $\varphi \in \mathcal{P}(X)$ and any $\mu \in \mathfrak{M}_F(X)$ put $S(\mu,\varphi) = \{x \in X \mid$ there exists a function $g \in U(X)$ such that

$$0 \leq g \leq \varphi^x \quad \text{and} \quad \mu(g) \neq 0\}.$$

Clearly, if $\varphi_1 \leq \varphi_2$ then $\varphi^x_1 \geq \varphi^x_2$ and $S(\mu, \varphi_1) \supset \supset S(\mu, \varphi_2)$. Put $S(\mu) = \bigcap_{\varphi \in \mathcal{P}(X)} S(\mu, \varphi)$.

Remark. Consider the associated Radon measure $\check{\mu}$ on the Samuel compactification $\check{X}$ of the space X [5]. It is easy to see that $S(\mu) = X \cap \operatorname{supp} \check{\mu}$.

3.2. Proposition. Let $\mu \in \mathcal{M}_F(X)$, $\varphi \in \mathcal{P}(X)$, $f \in U(X)$ and $f(x) = 0$ for any $x \in S(\mu, \varphi)$. Then $\mu(f) = 0$.

Proof. As $f = f^+ - f^-$ one can assume $f \geq 0$. As $\mu(f) = \lim_{n \to \infty} \mu(f \wedge n)$ one can assume f is bounded. Thus without any loss of generality we shall assume that $0 \leq f \leq 1$.

For any finite set $F \subset X \setminus S(\mu, \varphi)$ put $f_F = f \wedge (\max_{x \in F} \varphi^x)$. Order finite subsets of $X \setminus S(\mu, \varphi)$ by inclusion. Then $\lim_F f_F = f$ pointwise, the set $\{f_F \mid F \text{ finite} \subset X \setminus S(\mu, \varphi)\}$ is U.E., and hence $\mu(f) = \lim_F \mu(f_F)$. But for any finite set $F \subset X \setminus S(\mu, \varphi)$ one can write $f_F = \sum_{x \in F} f_x$ where $f_x \in U(X)$ and $0 \leq f_x \leq \varphi^x$ for $x \in F$.

Consequently $\mu(f_F) = 0$ for any finite set $F \subset X \setminus S(\mu, \varphi)$ and $\mu(f) = 0$,

Q.E.D.

3.3. Proposition. For any $\mu \in \mathcal{M}_F(X)$ we have $S(\mu) = \bigcap_{\varphi} \overline{S(\mu, \varphi)}$; consequently the set $S(\mu)$ is closed.

Proof. If $x \in X \setminus S(\mu, \varphi)$ and $\varphi(y,x) < \frac{1}{2}$ then $y \notin S(\mu, 2\varphi)$. Hence $S(\mu, \varphi) \supset \overline{S(\mu, 2\varphi)}$.

The following lemma shows that the set $S(\mu)$ supports the measure μ if the sets $S(\mu, \varphi)$ are not "too large". This helps to prove Theorem 4.2 below.

3.4. Lemma. Let X be a complete uniform space, let $\mu \in \mathcal{M}_F(X)$. Suppose that for any $\varphi \in \mathcal{P}(X)$ there exists a finite number of sets $R_i^{\wp} \subset X$, $i = 1,2,\dots,n(\wp)$, such that $\wp$-diam $R_i^{\wp} \leq 6$ for $i = 1,2,\dots,n(\wp)$, and $S(\mu,\wp) \subset \bigcup_{i=1}^{n(\wp)} R_i^{\wp}$. Then the set $S(\mu)$ is compact and the following holds: if $f \in U(X)$ and $f(x) = 0$ for each $x \in S(\mu)$ then $\mu(f) = 0$.

Proof. I. The set $S(\mu)$ is precompact, hence it is compact according to 3.3.

II. Suppose that $f \in U(X)$ and $f(x) = 0$ for $x \in S(\mu)$. Choose any $\varepsilon > 0$. I claim that there exists a pseudometric $\wp \in \mathcal{P}(X)$ such that $|f(x)| < \varepsilon$ for any $x \in S(\mu,\wp)$ (the claim is proved below). Put $g = (f^+ - \varepsilon)^+ - (f^- - \varepsilon)^+$: one has $\|f - g\| < \varepsilon$ and $g(x) = 0$ for any $x \in S(\mu,\wp)$.

Hence $|\mu(f)| \leq |\mu(g)| + |\mu(f-g)| \leq \varepsilon \|\mu\|$. As $\varepsilon > 0$ was arbitrary, the conclusion follows.

III. It remains to prove the claim stated above. Suppose it does not hold, i.e. there exists an $\varepsilon > 0$ such that $\tilde{S}_{\wp} = S(\mu,\wp) \cap \{x \in X \mid |f(x)| \geq \varepsilon\} \neq \emptyset$ for each $\wp \in \mathcal{P}(X)$. Then $\{\tilde{S}_{\wp} \mid \wp \in \mathcal{P}(X)\}$ is a base of a filter and there exists an ultrafilter $\mathcal{F}$ containing it. Now assumption in Lemma implies that for any $\wp \in \mathcal{P}(X)$ there is an $i(\wp)$ such that $R_{i(\wp)}^{\wp} \cap \{x \in X \mid |f(x)| \geq \varepsilon\} \in \mathcal{F}$. Hence $\mathcal{F}$ is a Cauchy filter and $\bigcap\{\overline{F} \mid F \in \mathcal{F}\} = \{x_0\}$; clearly $|f(x_0)| \geq \varepsilon$.

On the other hand, $x_0 \in S(\mu)$ and $f(x_0) = 0$.

This is the desired contradiction.

§ 4. Free uniform measures on sub-inversion-closed spaces.

The following property of sub-inversion-closed spaces is exactly what we need in the proof of Theorem 4.2 below.

4.1. Lemma. Given a sub-inversion-closed space X, a pseudometric $\varphi \in \mathcal{P}(X)$ and a countable set $Y \subset X$ such that $\varphi(x,y) \geq 3$ for $x,y \in Y$, $x \neq y$. Suppose further that for each $y \in Y$ we are given a function $f_y \in U(X)$ and a real number K_y such that $0 \leq f_y \leq K_y \cdot \varphi^y$. Then the function $\sum_{y \in Y} f_y$ is uniform on X.

Proof. Find an inversion-closed space Z such that X is a subspace of Z. f_y's and φ may be extended over Z: find $\tilde{f}_y \in U(Z)$ and $\tilde{\varphi} \in \mathcal{P}(Z)$ such that $\tilde{f}_y$ extends f_y for any $y \in Y$, $\tilde{\varphi}$ extends φ, and $0 \leq \tilde{f}_y \leq \tilde{\varphi}^y \cdot K_y$ for $y \in Y$ (this certainly can be done: if necessary, take $(\tilde{f}_y \wedge K_y \cdot \tilde{\varphi}^y)^+$ instead of $\tilde{f}_y$).

Then $\sum_{y \in Y} \tilde{f}_y$ is a Coz-function on an inversion-closed space Z, hence it is uniform and its restriction $\sum_{y \in Y} f_y$ is uniform on X,

Q.E.D.

4.2. Theorem. Let X be a complete sub-inversion-closed uniform space and let $\mu \in \mathcal{M}_F(X)$. Then there exists a compact set $C \subset X$ and a Radon measure m on C such that $\mu(f) = \int_C f dm$ for any $f \in U(X)$.

Proof. Put $C = S(\mu)$.

I. At first observe that the condition stated in 3.4 holds. Indeed, if it does not then there exists a pseudometric $\varphi \in \mathcal{P}(X)$ such that the set $S(\mu,\varphi)$ is not covered by any finite number of sets of φ-diameter ≤ 6. Hence one can inductively construct an infinite countable set $Y = \{y_1, y_2 \ldots \ldots\} \subset S(\mu,\varphi)$ such that $\varphi(y_k, y_\ell) \geq 3$ for $k \neq \ell$. For any $\ell = 1,2,\ldots$ there exists a function $g_\ell \in U(X)$ such that

$0 \leq g_\ell \leq \varphi^{y_\ell}$ and $\mu(g_\ell) \neq 0$. Choose real numbers K_ℓ, $\ell = 1,2,\ldots$, such that $|K_\ell \cdot \mu(g_\ell)| \geq \ell + \sum_{k=1}^{\ell-1} |K_k \cdot \mu(g_k)|$ and put $f_\ell = K_\ell \cdot g_\ell$, $f = \sum_{\ell=1}^{\infty} f_\ell$.

Lemma 4.1 implies that the set $\{\sum_{k=1}^{\ell} f_k \mid \ell = 1,2,\ldots\}$ is U.E., hence $\mu(f) = \lim_{\ell \to \infty} \mu(\sum_{k=1}^{\ell} f_k)$.

On the other hand, for $\ell = 1,2,\ldots$ we have

$$|\mu(\sum_{k=1}^{\ell} f_k)| \geq |K_\ell \cdot \mu(g_\ell)| - \sum_{k=1}^{\ell-1} |K_k \cdot \mu(g_k)| \geq \ell,$$

a contradiction.

II. Thus 3.4 applies and we have $\mu(f) = 0$ whenever $f(x) = 0$ for each $x \in C$.

For any $f \in U(X)$ denote $\tilde{f}$ its restriction to C: if $f, g \in U(X)$ and $\tilde{f} = \tilde{g}$ then $\mu(f) = \mu(g)$, hence the formula $\tilde{\mu}(\tilde{f}) = \mu(f)$ defines a continuous linear form on the Banach space $U_b(C)$ = the Banach space of all continuous functions on C. Consequently $\tilde{\mu}$ is represented by a Radon measure m on C, Q.E.D.

4.3. <u>Reformulation</u>. If X is any uniform space, denote by $\mathfrak{M}_C(X)$ the space of "Radon measures with a compact support in X": $\mu \in \mathfrak{M}_C(X)$ iff there exists a compact set $C \subset X$ and a Radon measure m on C such that $\mu(f) = \int_C f dm$ for any function $f \in U(X)$.

Now if X is any sub-inversion-closed space then the completion $\hat{X}$ of X is sub-inversion-closed as well and according to 4.2 we have $\mathfrak{M}_F(X) \cong \mathfrak{M}_F(\hat{X}) = \mathfrak{M}_C(\hat{X})$.

§ 5. <u>Order-bounded functionals</u>.

$\mathfrak{M}_{ob}(X)$ will denote the space of order-bounded linear functionals on the space U(X). Thus $\mu \in \mathfrak{M}_{ob}(X)$ iff for any

$f \in U(X)$, μ is bounded on the set $\{ g \in U(X) \mid |g| \leq f \}$. It is well-known ([17], V-1.1, 1.4) that $\mu \in \mathfrak{M}_{ob}(X)$ if and only if μ is a difference of two positive linear functionals on U(X). If this is so then $\mu = \mu^+ - \mu^-$ where $\mu^+(f) = \sup \{ \mu(g) \mid g \in U(X) \;\&\; 0 \leq g \leq f \}$ for $f \in U(X)$, $f \geq 0$.

It is readily seen that any element of $\mathfrak{M}_{ob}(X)$ is a measure in the sense of § 1.

5.1. Proposition. If $\mu \in \mathfrak{M}_F(X)$ is order-bounded then the linear functional μ^+ (defined by $\mu^+(f) = \sup \{ \mu(g) \mid 0 \leq g \leq f \;\&\; g \in U(X) \}$ for $f \in U(X)$, $f \geq 0$) belongs to the space $\mathfrak{M}_F(X)$.

Proof. See ([3], T.1).

5.2. Corollary. For any uniform space X, the inclusion $\mathfrak{M}_F(X) \subset \mathfrak{M}_{ob}(X)$ holds if and only if the space $\mathfrak{M}_F(X)$ is spanned by its positive cone.

Remark. If $\mathbb{R}$ denotes the real line with the usual uniformity then the space $\mathfrak{M}_F(\mathbb{R})$ is not included in $\mathfrak{M}_{ob}(\mathbb{R})$ ([15], 3.3).

On the other hand, for sub-inversion-closed spaces we have the following result:

5.3. Proposition. Let X be a sub-inversion-closed uniform space. Then $\mathfrak{M}_F(X) \subset \mathfrak{M}_{ob}(X)$ and the space $\mathfrak{M}_F(X)$ is spanned by its positive cone.

Proof. $\mathfrak{M}_F(X) \cong \mathfrak{M}_C(\hat{X})$ according to 4.3 and $\mathfrak{M}_C(\hat{X}) \subset \mathfrak{M}_{ob}(\hat{X}) \cong \mathfrak{M}_{ob}(X)$ obviously. Thus 5.2 applies.

§ 6. σ-additive functionals on U(X)

Denote by $\mathcal{M}_{\sigma\sigma}(X)$ the linear space of those linear functionals μ on the space U(X) that satisfy the following condition:

If $f_n \in U(X)$ for $n = 1,2,\dots$ and $f_n \searrow 0$ then $\lim_n \mu(f_n) = 0$.

6.1. Lemma. Let X be any uniform space, let $\mu \in \mathcal{M}_{\sigma\sigma}(X)$. Then:

a) for any $g \in U(X)$ it holds $\mu(g) = \lim_{n\to\infty} \mu(g \wedge n)$

b) μ is a measure.

Proof. a) is obvious.

As for b), assume that μ is not a measure in the sense of § 1, i.e. μ is not norm-continuous: for $n = 1,2,\dots$ there exist functions $g_n \in U_b(X)$ such that $\| g_n \| \le 1$ and $\mu(g_n) > 2^n$. As $g_n = g_n^+ - g_n^-$ one can assume $0 \le g_n \le 1$; if this is the case then the function $g = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n$ is uniform, $\sum_{n=1}^{N} \frac{1}{2^n} g_n \nearrow g$ as $N \to +\infty$ and $\mu\left(\sum_{n=1}^{N} \frac{1}{2^n} g_n\right) > N$, a contradiction.

6.2. Proposition. For any uniform space X we have $\mathcal{M}_{\sigma\sigma}(X) \subset \mathcal{M}_{ob}(X)$.

Proof. Assume $\mu \in \mathcal{M}_{\sigma\sigma}(X) \setminus \mathcal{M}_{ob}(X)$. Then there exists a function $f \in U(X)$ such that μ is not bounded on the set $\{g \in U(X) \mid |g| \le f\}$. Using the decomposition $g = g^+ - g^-$ and 6.1 (a) one sees that μ is not bounded on the set $\{g \in U_b(X) \mid 0 \le g \le f\}$.

Construct inductively functions $g_n \in U_b(X)$, $n = 0,1,\dots$, such that $g_0 = 0$ and $0 \le g_n \le f$, $|\mu(g_n)| > 2 \|\mu\| \cdot \|g_{n-1}\| + n$ for $n = 1,2,\dots$.

Put $h_n = g_n \vee (\|g_{n-1}\| \wedge f)$ for $n = 1,2,\dots$.

Then $h_n \in U_b(X)$ and $(f - h_n) \searrow 0$.

On the other hand, we shall see that $|\mu(h_n)| > n$ for $n = 1,2,\dots$ - this will be the contradiction.

In fact, one has $\mu(h_n) + \mu(g_n \wedge \|g_{n-1}\|) = \mu(g_n) + \mu(\|g_{n-1}\| \wedge f)$, hence $|\mu(h_n)| \geq |\mu(g_n)| - 2\|\mu\| \cdot \|g_{n-1}\| > n$ as claimed.

The proposition is proved.

For the converse inclusion, we must restrict ourselves; even the class of sub-inversion-closed spaces is too rich. However, for inversion-closed spaces it is true; in fact, the proof is well-known ([2], 3.1.1).

6.3. <u>Proposition.</u> If a space X is inversion-closed then $\mathfrak{M}_{ob}(X) \subset \mathfrak{M}_{\sigma\sigma}(X)$.

<u>Proof.</u> It suffices to show that $\mu \in \mathfrak{M}_{\sigma\sigma}(X)$ whenever $\mu \in U(X)^*$ and $\mu \geq 0$ - let it be the case. Choose $f_n \searrow 0$ and $\varepsilon > 0$.

The sequence of Coz-sets $\{x \in X \mid f_n(x) < \varepsilon\}$, $n = 1,2,\dots$, covers X. Hence the sum $f = \sum_{n=1}^{\infty} (f_n - \varepsilon)^+$ is finite.

Consider any $a,b \in \mathbb{R}$, $a < b$:

then $\{x \in X \mid f(x) > a\} = \bigcup_{k=1}^{\infty} \{x \in X \mid \sum_{n=1}^{k} (f_n(x) - \varepsilon)^+ > a\}$ is a Coz-set and $\{x \in X \mid f(x) < b\} = \bigcup_{k=1}^{\infty} \{x \in X \mid f_k(x) < \varepsilon \;\&\; \sum_{n=1}^{\infty} (f_n(x) - \varepsilon)^+ < b\} = \bigcup_{k=1}^{\infty} \{x \in X \mid f_k(x) < \varepsilon \;\&\; \sum_{n=1}^{k} (f_n(x) - \varepsilon)^+ < b\}$ is a Coz-set as well.

Thus f is a Coz-function on an inversion-closed space and $f \in U(X)$. Consequently $\lim_{n\to\infty} \mu((f_n - \varepsilon)^+) = 0$ and as $\mu(f_n) \leq \varepsilon \cdot \mu(1) + \mu((f_n - \varepsilon)^+)$ and $\varepsilon > 0$ was arbitrary, we get $\lim_n \mu(f_n) = 0$, Q.E.D.

6.4. Let me sum up for the later use:

Theorem. For any inversion-closed space X we have $\mathfrak{M}_C(\hat{X}) \cong \mathfrak{M}_F(X) \subset \mathfrak{M}_{ob}(X) = \mathfrak{M}_{\sigma\sigma}(X)$.

6.5. Remark. The inclusion $\mathfrak{M}_F(X) \subset \mathfrak{M}_{\sigma\sigma}(X)$ for inversion-closed spaces can be proved directly by the method of the proof of 4.2 in [15], using Theorem from § 2 above.

§ 7. Riesz and separable Riesz measures

Let us begin with the following lemma.

7.1. Lemma (cf.[9], § 5). Let X be a uniform space such that countable uniform covers form a basis of its uniform covers. Then $\mathfrak{M}_{\sigma\sigma}(X) \subset \mathfrak{M}_F(X)$.

Proof. Let $\mu \in \mathfrak{M}_{\sigma\sigma}(X)$. Then $\mu = \mu^+ - \mu^-$ and standard argument shows that $\mu^+, \mu^- \in \mathfrak{M}_{\sigma\sigma}(X)$; hence we can and shall assume that $\mu \geq 0$. Let $\{f_\alpha\}_{\alpha\in A}$ be a net such that the set $\{f_\alpha \mid \alpha \in A\}$ is U.E. and $\lim f_\alpha = 0$ pointwise. One must prove that $\lim \mu(f_\alpha) = 0$.

Put $g_\alpha = \sup_{\beta \geq \alpha} |f_\beta|$ for any $\alpha \in A$; the set $\{g_\alpha \mid \alpha \in A\}$ is U.E. and $g_\alpha \searrow 0$.

It follows from the assumption that there exists a countable set $D \subset X$ such that

$$(*) \quad \forall \varepsilon > 0 \quad \forall x \in X \quad \exists d \in D \quad \forall \alpha \in A \; |g_\alpha(x) - g_\alpha(d)| < \varepsilon .$$

By diagonal method one finds an increasing sequence $\alpha(n)$ of indices such that $\lim_{n\to\infty} g_{\alpha(n)}(d) = 0$ for any $d \in D$. Now $(*)$ implies that $g_{\alpha(n)} \searrow 0$ for $n \longrightarrow \infty$ and $\lim_{n\to\infty} \mu(g_{\alpha(n)}) = 0$ because μ is σ-additive.

Hence $\lim_\alpha |\mu(f_\alpha)| \leq \lim_\alpha \mu(|f_\alpha|) = 0$, Q.E.D.

Now we are going to see howe the results of preceding paragraphs yield known facts for the space of Riesz measures, resp. separable Riesz measures (denoted M_{ν} , resp. M by French authors and M_C, resp. M_{SC} by Kirk).

Besides free uniform measures we shall need here so called uniform measures (see e.g. [4],[15]). Below I use the canonical one-to-one map r_X: $\mathfrak{M}_F(X) \longrightarrow \mathfrak{M}_U(X)$; its properties are described in [15].

7.2. Notation. Given a Hausdorff completely regular topological space T, consider two uniformities on the underlying set: t_fT is the fine uniform space associated with T (t_fT is the finest uniformity agreeing with the topology of T), cT denotes the uniform space projectively generated by all real-valued functions continuous on T (cT has the coarsest uniformity such that all functions continuous on T are uniform).

One has $U(t_fT) = U(cT)$ = the space of real-valued functions continuous on T, and consequently both the uniform spaces t_fT and cT are inversion-closed.

The elements of the space $\mathfrak{M}_U(t_fT)$ are called separable measures on T (see e.g. [18]). The elements of the space $\mathfrak{M}_{ob}(t_fT) = \mathfrak{M}_{ob}(cT)$ are called Riesz measures on T by Berruyer and Ivol [2].

7.3. Riesz measures. Let me notice that $\widehat{cT}$ is just the Hewitt realcompactification of the space T; 6.4 and 7.1 yield the equalities

$\mathfrak{M}_{ob}(cT) = \mathfrak{M}_{\sigma\sigma}(cT) = \mathfrak{M}_F(cT) \cong \mathfrak{M}_C(\widehat{cT})$ (see [2], 3.1 and [11], T. 14, 17).

7.4. As for the space t_fT we get the following result:

Proposition. Let T be any Hausdorff completely regular space. Then

a) [10] We have $\mathfrak{M}_F(t_f T) \cong \mathfrak{M}_C(\widehat{t_f T})$;

b) ([13], 9.4) Free uniform measures on the space $t_f T$ are just the separable Riesz measures on T.

More exactly: Consider the canonical one-to-one maps in the commuting diagram

$$\begin{array}{ccc} \mathfrak{M}_U(t_f T) & \longrightarrow & \mathfrak{M}_U(cT) \\ \uparrow r_{t_f T} & & \uparrow r_{cT} \\ \mathfrak{M}_F(t_f T) & \longrightarrow & \mathfrak{M}_F(cT) \end{array}$$

(horizontal arrows are induced by the identity map $t_f T \longrightarrow$ $\longrightarrow cT$).
Identify the spaces $\mathfrak{M}_U(t_f T)$, $\mathfrak{M}_F(t_f T)$ and $\mathfrak{M}_F(cT)$ with linear subspaces of $\mathfrak{M}_U(cT)$ by means of these maps. Then $\mathfrak{M}_F(t_f T) = \mathfrak{M}_U(t_f T) \cap \mathfrak{M}_F(cT)$.

Proof. a) follows from 4.3.

b) Obviously $\mathfrak{M}_F(t_f T) \subset \mathfrak{M}_U(t_f T) \cap \mathfrak{M}_F(cT)$. Conversely, if $\mu \in \mathfrak{M}_U(t_f T) \cap \mathfrak{M}_F(cT)$ then $\mu \in \mathfrak{M}_U(t_f T)$ and finite $\lim_{M \to \infty} \mu((-M) \vee (f \wedge M))$ exists for any $f \in U(cT) = U(t_f T)$; ([15], 4.5) implies that $\mu \in \mathfrak{M}_F(t_f T)$.

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