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EDZ-VARIETIES: THE SCHREIER PROPERTY AND EPIMORPHISMS ONTO

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Abstract: By an EDZ-variety we mean a variety of universal algebras with equationally definable zeros. The present paper is a continuation of [6] where EDZ-varieties were studied. We shall find a necessary and sufficient condition for an EDZ-variety to have the Schreier property. Further we shall prove that in any EDZ-variety epimorphisms are just onto homomorphisms; this is rather unexpected, since there are many EDZ-varieties without the amalgamation property.

Key Words: Variety, universal algebra, zero element, free algebra, Schreier property, epimorphism.

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1. Introduction and preliminaries. To be able to make a picture of possible interconnections between various properties of varieties of universal algebras, one must have a suitable supply of examples. The system of EDZ-varieties, investigated in [6], proved to be convenient, since it is sufficiently broad and for many general properties P the problem of deciding which EDZ-varieties have P is algorithmically solvable. In [6] we were concerned namely with the following properties: the amalgamation property; the strong amalgamation property; having enough subdirectly irreducible members; having enough simple members; having few

simple members. The present paper is a continuation of [6]; we shall be concerned with the Schreier property and with weak forms of the strong amalgamation property.

There are many papers on the Schreier property; let us mention e.g. [1],[2],[4],[5],[7]. In [3] it is proved that uncountably many minimal varieties of 2-unary algebras have the Schreier property.

The terminology and notation used in this paper are those of [6]; however, it will be convenient to have a summary of the basic concepts concerning EDZ-varieties at hand.

X is a fixed infinite countable set of symbols, called the variables. W_Δ is the algebra of Δ-terms, i.e. of formal expressions formed from variables and symbols from Δ. It is an absolutely free Δ-algebra over X.

If $u,v \in W_\Delta$, we write $u \leq v$ iff f(u) is a subterm of v for some endomorphism f of W_Δ. For every subset J of W_Δ we put $\Phi(J) = \{u \in W_\Delta ; v \leq u \text{ for some } v \in J\}$. If $J \subseteq W_\Delta$, then $\mathcal{Z}_J$ denotes the variety of Δ-algebras satisfying any identity $u \mathrel{\hat{=}} v$ such that $u,v \in \Phi(J)$. A variety K is called an EDZ-variety if $K = \mathcal{Z}_J$ for some $J \subseteq W_\Delta$. If J is non-empty, then a Δ-algebra A belongs to $\mathcal{Z}_J$ iff A has a zero element 0_A and $f(t) = 0_A$ for every $t \in J$ and every homomorphism $f: W_\Delta \longrightarrow A$. If u,v are two Δ-terms, then the identity $u \mathrel{\hat{=}} v$ is satisfied in $\mathcal{Z}_J$ iff either $u = v$ or $u,v \in \Phi(J)$. Every EDZ-variety K can be expressed in the form $K = \mathcal{Z}_J$, where J is an irreducible set of Δ-terms, i.e. a set $J \subseteq W_\Delta$ such that $u,v \in J$ and $u \leq v$ imply $u = v$.

2. EDZ-varieties with the Schreier property. A variety K is said to have the Schreier property if any non-trivial subalgebra (i.e. any subalgebra with at least two elements) of any K-free algebra is K-free. In this section we shall find a necessary and sufficient condition for an arbitrary EDZ-variety to have the Schreier property.

Let a type Δ be given. For every non-empty subset J of W_Δ we define a Δ-algebra W_J as follows: its underlying set is the set $(W_\Delta \setminus \Phi(J)) \cup \{0\}$; if $F \in \Delta$, $t_1,\dots,t_{n_F} \in W_\Delta \setminus \Phi(J)$ and $F(t_1,\dots,t_{n_F}) \notin \Phi(J)$, then we put $F_{W_J}(t_1,\dots,t_{n_F}) = F(t_1,\dots,t_{n_F})$; if $F \in \Delta$, $p_1,\dots,p_{n_F} \in W_J$ and if $F_{W_J}(p_1,\dots,p_{n_F})$ is not yet defined, then we put $F_{W_J}(p_1,\dots,p_{n_F}) = 0$. It is easy to see that W_J is the $\mathfrak{X}_J$-free algebra over X.

2.1. Lemma. Let J be a subset of W_Δ such that $\Phi(J) \neq W_\Delta$. Let Y be a non-empty set and let H be a $\mathfrak{X}_J$-free algebra over Y. Then

(1) Y is just the set of irreducible elements of H;

(2) if $F, G \in \Delta$, $a_1,\dots,a_{n_F}$, $b_1,\dots,b_{n_G} \in H$ and $F_H(a_1,\dots,a_{n_F}) = G_H(b_1,\dots,b_{n_G}) \neq 0_H$, then $F = G$ and $a_1 = b_1,\dots,a_{n_F} = b_{n_F}$;

(3) if A is a subalgebra of H, then A is generated by the set of irreducible elements of A.

Proof is easy.

2.2. Lemma. Let J be a subset of W_Δ such that $\Phi(J) \neq W_\Delta$. The variety $\mathfrak{X}_J$ has the Schreier property iff any non-trivial subalgebra of a $\mathfrak{X}_J$-free algebra over an infinite countable set is $\mathfrak{X}_J$-free.

Proof. The direct implication is immediate. Let us prove the converse. Since $\mathfrak{X}_J$-free algebras over finite sets are subalgebras of $\mathfrak{X}_J$-free algebras over an infinite countable set, it is enough to prove that if A is a non-trivial subalgebra of a $\mathfrak{X}_J$-free algebra H over an uncountable set Y, then A is $\mathfrak{X}_J$-free. For every non-empty subset M of H denote by $[M]$ the subalgebra of H generated by M. Let us call a subset M independent if $[M]$ is a $\mathfrak{X}_J$-free algebra over M. By 2.1 it is sufficient to prove that the set D of irreducible elements of A is independent. Evidently, D will be independent if we prove that every finite $E \subseteq D$ with Card $[E] \geq 2$ is independent. Let E be such a finite subset. There exists an infinite countable set $Y' \subseteq Y$ such that $E \subseteq [Y']$. Since $A \cap [Y']$ is a non-trivial subalgebra of $[Y']$ and $[Y']$ is $\mathfrak{X}_J$-free over Y', the algebra $A \cap [Y']$ is $\mathfrak{X}_J$-free over the set E' of irreducible elements of $A \cap [Y']$. Every element of E is irreducible in A and thus irreducible in $A \cap [Y']$, too, so that $E \subseteq E'$. Since E' is independent, E is independent, too.

The author does not know whether 2.2 is true for arbitrary varieties.

2.3. Theorem. Let J be an irreducible set of Δ-terms. The following two conditions are equivalent:

(1) the variety $\mathfrak{X}_J$ has the Schreier property;

(2) if $F(t_1,\dots,t_{n_F}) \in J$ for some $F \in \Delta$ and some terms $t_1,\dots,t_{n_F}$, then every variable occurring in $F(t_1,\dots,t_{n_F})$

belongs to $\{t_1,\dots,t_{n_F}\}$.

Proof. We shall suppose that J is non-empty and $\Phi(J) \neq W_\Delta$, since otherwise everything is clear.

(1) $\Longrightarrow$ (2): Let $\mathfrak{X}_J$ have the Schreier property and let $F(t_1,\dots,t_{n_F}) \in J$. Put $n = n_F$ and let $n \geq 1$. Denote by A the subalgebra of W_J generated by $\{t_1,\dots,t_n\}$ and denote by I the set of irreducible elements of A. Evidently $I \subseteq \{t_1,\dots,t_n\}$. Since $\mathfrak{X}_J$ has the Schreier property, A is $\mathfrak{X}_J$-free over I and thus there exists a homomorphism $f: A \longrightarrow W_J$ such that the restriction $f \upharpoonright I$ is an injective mapping of I into X. As W_Δ is absolutely free, there exists an endomorphism g of W_Δ such that $g(f(t)) = t$ for all $t \in I$.

Let us prove by the induction on the length of u that if $u \in A$ and $u \neq 0$, then $f(u) \neq 0$ and $g(f(u)) = u$. If us is a variable, then $u \in I$ and $g(f(u)) = u$ follows immediately from the definition of g. Let $u = G(u_1,\dots,u_{n_G})$. If $n_G = 0$, then $f(u) = g(u) = u$. Let $n_G \geq 1$. If one of the elements $u_1,\dots,u_{n_G}$ does not belong to A, then evidently $u \in I$ and, again, $g(f(u)) = u$ follows from the definition of g. Let $\{u_1,\dots,u_{n_G}\} \subseteq A$. Of course, $u_1,\dots,u_{n_G} \neq 0$. By the induction $f(u_1) \neq 0,\dots,f(u_{n_G}) \neq 0$ and $g(f(u_1)) = u_1,\dots,g(f(u_{n_G})) = u_{n_G}$. Suppose $f(u) = 0$. Then $G_{W_J}(f(u_1),\dots,f(u_{n_G})) = f(G_A(u_1,\dots,u_{n_G})) = f(u) = 0$, so that $G(f(u_1),\dots,f(u_{n_G})) \in \Phi(J)$. Consequently, $g(G(f(u_1),\dots,f(u_{n_G}))) \in \Phi(J)$, i.e. $G(u_1,\dots,u_{n_G}) \in \Phi(J)$, i.e. $u \in \Phi(J)$, a contradiction. Hence $f(u) \neq 0$. We have $g(f(u)) =$

$= g(G_{W_J}(f(u_1),\dots,f(u_{n_G}))) = g(G(f(u_1),\dots,f(u_{n_G}))) =$
$= G(g(f(u_1)),\dots,g(f(u_{n_G}))) = G(u_1,\dots,u_{n_G}) = u.$

Especially $f(t_1) \neq 0,\dots,f(t_n) \neq 0$ and $g(f(t_1)) = t_1,\dots,g(f(t_n)) = t_n$. We have $F_{W_J}(f(t_1),\dots,f(t_n)) = f(F_A(t_1,\dots,t_n)) = f(F_{W_J}(t_1,\dots,t_n)) = f(0) = 0$, so that $F(f(t_1),\dots,f(t_n)) \in \Phi(J)$. Since $F(t_1,\dots,t_n) \in J$ and $g(F(f(t_1),\dots,f(t_n))) = F(g(f(t_1)),\dots,g(f(t_n))) = F(t_1,\dots,t_n)$, it follows from the irreducibility of J that the terms $F(t_1,\dots,t_n)$ and $F(f(t_1),\dots,f(t_n))$ are similer. Consequently, every element of I is a variable.

Thus I is a set of variables and A is the sublagebra of W_J generated by I. Since every t_i belongs to A, every variable occurring in t belongs to $I \subseteq \{t_1,\dots,t_n\}$.

(2) $\Longrightarrow$ (1): By 2.2 it is enough to prove that if A is a non-trivial subalgebra of W_J, then A is $\mathfrak{Z}_J$-free. Denote by I the set of irreducible elements of A. By 2.1, A is generated by I. Let $B \in \mathfrak{Z}_J$ and let f be a mapping of I into B. Define a mapping h of A into B as follows: $h(0_A) = 0_B$; if $a \in I$ then $h(a) = f(a)$; if $a \in A$, $a \neq 0_A$ and $a = F_A(a_1,\dots,a_{n_F})$, then $h(a) = F_B(h(a_1),\dots,h(a_{n_F}))$. It is enough to prove that h is a homomorphism of A into B, and for this it is sufficient to show that if $F \in \Delta$, $a_1,\dots,a_{n_F} \in A \setminus \{0_A\}$ and $F(a_1,\dots,a_{n_F}) \in \Phi(J)$, then $F_B(h(a_1),\dots,h(a_{n_F})) = 0_B$. There exists a term $t \in J$ with $t \leq F(a_1,\dots,a_{n_F})$. Evidently $F(a_1,\dots,a_{n_F}) = e(t)$ for some endomorphism e of W_Δ and there exist terms $t_1,\dots,t_{n_F}$ such that $t = F(t_1,\dots,t_{n_F})$ and $a_1 =$

$= e(t_1),\dots,a_{n_F} = e(t_{n_F})$. There exists a homomorphism $g:$ $: W_J \longrightarrow B$ such that if x is a variable and $x = t_i$ for some $i \in \{1,\dots,n_F\}$, then $g(x) = h(a_i)$. Let us prove by the induction on the length of u that if u is a subterm of at least one of the terms $t_1,\dots,t_{n_F}$, then $e(u) \in A$ and $g(u) =$ $= h(e(u))$. If u is a variable then by (2) we have $u = t_i$ for some $i \in \{1,\dots,n_F\}$ and thus $g(u) = h(e(u))$ follows from the defining property of g. Let $u = G(u_1,\dots,u_{n_G})$. By the induction assumption

$e(u) = G(e(u_1),\dots,e(u_{n_G})) \in A$ and $g(u) = G_B(g(u_1),\dots$ $\dots,g(u_{n_G})) = G_B(h(e(u_1)),\dots,h(e(u_{n_G}))) = h(e(u))$.

From this we get $F_B(h(a_1),\dots,h(a_{n_F})) = F_B(h(e(t_1)),\dots$ $\dots,h(e(t_{n_F}))) = F_B(g(t_1),\dots,g(t_{n_F})) = g(F(t_1,\dots,t_{n_F})) =$ $= g(t) = 0_B$, since $t \in J$.

2.4. *Corollary.* Let Δ be a type containing at least one at least binary symbol. Then for every proper (i.e. different from the variety of all Δ-algebras) EDZ-variety K of Δ-algebras there exist two proper EDZ-varieties L_1, L_2 such that $K \subset L_1$, $K \subset L_2$, L_1 has the Schreier property and L_2 has not the Schreier property. There exists an infinite increasing sequence of varieties of groupoids such that the varieties with odd indexes have and the varieties with even indexes have not the Schreier property.

3. *Weak forms of the strong amalgamation property.* Consider the following six conditions for a variety K:

(1) K has the strong amalgamation property;

(2) for every triple $A,B,C \in K$ such that A is a subalgebra of both B and C, $A = B \cap C$ and such that B,C are isomorphic over A there exists an algebra $D \in K$ such that both B and C are subalgebras of D;

(3) every monomorphism of the category K is an equalizer of a pair of K-morphisms;

(3′) if $B \in K$ and if A is a subalgebra of B, then there exists an algebra $C \in K$ and two homomorphisms $f,g\colon B \longrightarrow C$ such that $A = \{b \in B;\ f(b) = g(b)\}$;

(4) every epimorphism of the category K is a homomorphism onto;

(4′) if $B \in K$ and if A is a proper subalgebra of B, then there exists an algebra $C \in K$ and two homomorphisms $f,g\colon B \longrightarrow C$ such that $A \subseteq \{b \in B;\ f(b) = g(b)\} \neq B$.

It is easy to see that

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (3') \Longrightarrow (4) \Longleftrightarrow (4').$$

In [6] we have characterized EDZ-varieties with the strong amalgamation property. Now we shall prove

3.1. <u>Theorem</u>. Every EDZ-variety satisfies the condition (2).

<u>Proof</u>. We must prove that if J is a non-empty subset of W_Δ then the variety $\mathfrak{Z}_J$ satisfies (2). Let $A,B,C \in \mathfrak{Z}_J$, let A be a common subalgebra of B,C, let $A = B \cap C$ and let f be an isomorphism of B onto C such that $f(a) = a$ for all $a \in A$. Define a Δ-algebra D as follows: its underlying set is the set $B \cup C$; if $F \in \Delta$ and $a_1,\dots,a_{n_F} \in B$, put $F_D(a_1,\dots,a_{n_F}) = F_B(a_1,\dots,a_{n_F})$; if $a_1,\dots,a_{n_F} \in C$, put $F_D(a_1,\dots,a_{n_F}) =$

$= F_C(a_1,\dots,a_{n_F})$; if $a_1,\dots,a_{n_F} \in D$ and $\{a_1,\dots,a_{n_F}\}$ is a subset of neither B nor C, put $F_D(a_1,\dots,a_{n_F}) = 0_A$. It is enough to show that $D \in \mathfrak{Z}_J$. Let g be a homomorphism of W_Δ into D. There exists precisely one homomorphism $h: W_\Delta \longrightarrow B$ such that if x is a variable, then $h(x) = g(x)$ in the case $g(x) \in B$ and $h(x) = f^{-1}(g(x))$ in the case $g(x) \in C$. Let us prove by the induction on the length of t that if $t \in W_\Delta$, then $g(t) \in \{h(t), f(h(t)), 0_A\}$. If f is a variable, this follows from the definition of h. Let $t = F(t_1,\dots,t_{n_F})$, so that $g(t) = F_D(g(t_1),\dots,g(t_{n_F}))$. Using the induction assumption it is obvious that if $g(t_1) = h(t_1),\dots,g(t_{n_F}) = h(t_{n_F})$, then $g(t) = h(t)$ and if $g(t_1) = f(h(t_1)),\dots,g(t_{n_F}) = f(h(t_{n_F}))$, then $g(t) = f(h(t))$; in all other cases $g(t) = 0_A$.

If $t \in J$, then $h(t) = f(h(t)) = 0_A$ and thus $g(t) = 0_A = 0_D$. Since g was an arbitrary homomorphism of W_Δ into D, we get $D \in \mathfrak{Z}_J$.

3.2. Corollary. In any EDZ-variety epimorphisms are onto homomorphisms.

The variety of groupoids determined by

$$(xx.y)z = z(xx.y) = xx.y$$

is (by 3.2 and by Theorem 6.1 of [6]) an example of a variety which does not satisfy the amalgamation property but in which epimorphisms are onto homomorphisms.

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