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STRUCTURE OF TRIABELIAN QUASIGROUPS

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Abstract: A quasigroup is called triabelian if every
its subquasigroup which is generated by at most three ele-
ments is abelian. In the present paper, some basic struc-
tural theorems on triabelian quasigroups are proved.

Key Words: Quasigroup, Moufang loop.

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As it is well known, the class of distributive quasi-
groups has a large number of nice algebraic properties. It
is the purpose of this paper to show that the structure of
triabelian quasigroups is very similar to that of distribu-
tive quasigroups. In certain sense, this paper is a continu-
ation of the last section from [1]. First we recall some de-
finitions . A quasigroup Q is called an RF-quasigroup (LF-
quasigroup) if it satisfies the identity $bc.a. = bf(a).ca$
$(a.bc = ab.e(a)c)$, where $f(a)$ and $e(a)$ is the left and the
right local unit of a, resp. It is called an F-quasigroup
if it is both an LF and RF-quasigroup. Further, a quasi-
group Q is said to be a WA-quasigroup if $aa.bc = ab.ac$ and
$bc.aa = ba.ca$ for all $a,b,c \in Q$. If moreover $ab.ca = ac.ba$
then we shall say that Q is a WAD-quasigroup. Finally, an

abelian quasigroup is a quasigroup satisfying the identity $ab.cd = ac.bd$. Let Q be a quasigroup and $x \in Q$. Then L_x and R_x is the left and the right translation by x, resp. If Q is a commutative Moufang loop, then j is the identity of Q, $N(Q)$ is the nucleous of Q and a mapping g of Q into Q is said to be nuclear if $x^{-1}.g(x) \in N(Q)$ for each $x \in Q$.

The following lemma is implicitly contained in [2].

Lemma 1. Let Q be a commutative loop and h be a mapping of Q into Q. Then the following are equivalent:

(i) $(a.h(a))(bc) = (ab)(h(a)c)$ for all $a,b,c \in Q$.

(ii) Q is a Moufang loop and h is nuclear.

Theorem 1. The following conditions are equivalent for every quasigroup Q:

(i) Q is a WA-quasigroup and there exists $a \in Q$ such that $ab.ca = ac.ba$ for all $b,c \in Q$.

(ii) Q is a WA-quasigroup and Q is isotopic to a commutative Moufang loop.

(iii) Q is a WA-quasigroup and Q is isotopic to a Moufang loop.

(iv)   There are a commutative Moufang lopp $Q(\circ)$, $g,h \in$ $\in \operatorname{Aut} Q(\circ)$ and $x \in Q$ such that $gh = hg$, gh^{-1} is nuclear and $ab = (g(a) \circ h(b)) \circ x$ for all $a,b \in Q$.

(v) Q is a WAD-quasigroup.

Proof. (i) implies (ii). If $b,c \in Q$ then $(aa.ab)(ac.aa) =$
$= (aa.ab)(aa.ca) = (aa.aa)(ab.ca) = (aa.aa)(ac.ba) =$
$= (aa.ac)(aa.ba) = (aa.ac)(ab.aa)$. Hence $(aa.x)(y.aa) =$
$= (aa.y)(x.aa)$ for all $x,y \in Q$ and we can use [1, Proposition 4.8] and Lemma 1.

(iii) implies (iv). Let $x \in Q$ and $a \circ b = R_{xx}^{-1}(a) \cdot L_{xx}^{-1}(b)$
for all $a, b \in Q$.

As it is proved in [1], Propositions 4.1 and 4.8, $Q(\circ)$ is
a CI-loop. However, $Q(\circ)$ is a Moufang loop, and hence it
is commutative. The rest follows from [1, Proposition 4.8
and Theorem 4.9].

(iv) implies (v). Since gh^{-1} is a nuclear mapping and
$gh = hg$, $g^2 h^{-2} = gh^{-1} gh^{-1}$ is nuclear. According to Lemma 1,
$ab.ca = (((g^2(a) \circ gh(b)) \circ g(x)) \circ ((hg(c) \circ h^2(a)) \circ h(x))) \circ x =$
$= (((g^2(a) \circ gh(b))) \circ (gh(c) \circ h^2(a))) \circ (g(x) \circ h(x))) \circ x =$
$= (((g^2(a) \circ gh(c))) \circ g(x)) \circ ((hg(b) \circ h^2(a)) \circ h(x))) \circ x =$
$= ac.ba$ for all $a, b, c \in Q$.

Let Q be a WAD-quasigroup. A tetrad $(Q(\circ), g, h, x)$ is
called an arithmetical form of Q if the condition (iv) from
Theorem 1 is satisfied.

The following lemma is implicitly proved in [1], Theo-
rem 4.9.

Lemma 2. Let Q be a WAD-quasigroup and $x \in Q$. Then the-
re exists an arithemtical form $(Q(\circ), g, h, y)$ of Q such that
$xx.xx = j$.

Lemma 3. Let Q be a WAD-quasigroup with an arithmeti-
cal form $(Q(\circ), g, h, x)$. Then $x \in N(Q(\circ))$ and $a \circ g(a)$, $a \circ h(a) \in$
$\in N(Q(\circ))$ for every $a \in Q$, provided at least one of the fol-
lowing conditions holds:

(i) Q is an LF-quasigroup.

(ii) Q is an RF-quasigroup.

(iii) $(a.aa)(bc) = (ab)(aa.c)$ for all $a, b, c \in Q$.

(iv) $(bc)(aa.a) = (b.aa)(ca)$ for all $a, b, c \in Q$.

<u>Proof.</u> (i) As it is easy to see, $he(a) =$
$= (a \circ x^{-1}) \circ g(a^{-1})$ for each $a \in Q$. Since Q is an LF-quasi-
group,
$$(g(a) \circ ((hg(b) \circ h^2(c)) \circ h(x))) \circ x = a.bc = ab.e(a)c =$$
$$= (((g^2(a) \circ gh(b)) \circ g(x)) \circ ((((g(a) \circ g(x^{-1})) \circ g^2(a^{-1})) \circ$$
$$\circ h^2(x)) \circ h(c))) \circ x,$$
and hence
(1) $a \circ ((b \circ c) \circ h(x)) = ((g(a) \circ b) \circ g(x)) \circ ((((a \circ g(x^{-1})) \circ$
$\circ g(a^{-1})) \circ c) \circ h(x))$

for all $a,b,c, \in Q$. If we substitute $a = j$ in (1), we obtain
the equality
$$h(x) \circ (b \circ c) = (b \circ g(x)) \circ ((c \circ g(x^{-1})) \circ h(x)).$$
Multiplying the last equality by $h(x^{-1}) \circ g(x)$ and taking in-
to account that this element belongs to $N(Q(\circ))$ (since $gh =$
$= hg$ and gh^{-1} is nuclear), we get the equality $(b \circ c) \circ g(x) =$
$= (b \circ g(x)) \circ c$ for all $b,c \in Q$. Thus $g(x) \in N(Q(\circ))$, and con-
sequently $h(x), x \in N(Q(\circ))$. Now the equality (1) yields
$a \circ (b \circ c) = (g(a) \circ b) \circ ((a \circ g(a^{-1})) \circ c)$ and
$(a \circ b) \circ ((a^{-1} \circ g^{-1}(a)) \circ c) = g^{-1}(a) \circ (b \circ c) =$
$= (a \circ (a^{-1} \circ g^{-1}(a)) \circ (b \circ c)$ for all $a,b,c \in Q$. By Lemma 1, $a^{-1} \circ$
$a^{-1} \circ g^{-1}(a) \in N(Q(\circ))$. However $a \circ a \circ a \in N(Q(\circ))$ (since $Q(\circ)$
is a commutative Moufang loop), so that $a \circ g^{-1}(a) \in N(Q(\circ))$.
As $N(Q(\circ))$ is invariant under automorphisms, $a \circ g(a) \in$
$\in N(Q(\circ))$. Finally, $a^{-1} \circ gh^{-1}(a)$ is contained in $N(Q(\circ))$,
and therefore $g(a) \circ gh^{-1}(a) \in N(Q(\circ))$, $a \circ h^{-1}(a) \in N(Q(\circ))$
and $a \circ h(a) \in N(Q(\circ))$.

(ii) Similarly as for (i).

(iii) After some arrangements (using Lemma 1 and the
fact that gh^{-1} is nuclear), we can write the identity

$(a.aa)(bc) = (ab)(aa.c)$ as $(a \circ h((a \circ hg^{-1}(a)) \circ g(x))) \circ$
$\circ (b \circ c) = (a \circ b) \circ (h((a \circ hg^{-1}(a)) \circ g(x)) \circ c)$.
If $a = j$ then $hg(x) \circ (b \circ c) = b \circ (hg(x) \circ c)$, and hence
$x \in N(Q(\circ))$. Then $(a \circ h(a) \circ hg^{-1}(a))) \circ (b \circ c) =$
$= (a \circ b) \circ ((h(a \circ hg^{-1}(a) \circ c)$ and $a^{-1} \circ h(a \circ hg^{-1}(a)) \in$
$\in N(Q(\circ))$ by Lemma 1. However, gh^{-1} is nuclear, therefo-
re hg^{-1} is so and $h(a \circ hg^{-1}(a^{-1})) \in N(Q(\circ))$. Thus
$a^{-1} \circ h(a \circ a) \in N(Q(\circ))$. Finally, $h(a^{-1} \circ a^{-1} \circ a^{-1}) \in N(Q(\circ))$,
so that $a \circ h(a) \in N(Q(\circ))$. Similarly as in the proof of (i),
we can show that $a \circ g(a) \in N(Q(\circ))$.

(iv) Similarly as for (iii).

Lemma 4. Let a WAD-quasigroup Q have an arithmetical
form $(Q(\circ), g, h, x)$ such that $x \in N(Q(\circ))$ and $a \circ g(a)$,
$a \circ h(a) \in N(Q(\circ))$ for every $a \in Q$. Then

(i) Q is an F-quasigroup.

(ii) If $a, b, c, d \in Q$ and $ab.cd = ac.bd$ then $ab.(c(dd.dd)) =$
$ac.(b(dd.dd))$.

Proof. (i) It is an easy exercise.

(ii) Since $ab.cd = ac.bd$ and $x \in N(Q(\circ))$,

(2) $(g^2(a) \circ gh(b)) \circ (hg(c) \circ h^2(d)) = (g^2(a) \circ gh(c)) \circ$
$\circ (hg(b) \circ h^2(d))$.

Put $u = (g^2(d) \circ h^2(d)) \circ (hg(d) \circ hg(d))$. We shall prove that
$d^{-1} \circ u \in N(Q(\circ))$. Indeed, $d^{-1} \circ g(d^{-1})$, $g(d) \circ gh(d)$,
$g(d) \circ g^2(d)$, $d^{-1} \circ h(d^{-1})$ and $h(d) \circ h^2(d)$ belong to $N(Q(\circ))$.
Hence $d^{-1} \circ g^2(d)$, $d^{-1} \circ gh(d)$ and $d^{-1} \circ h^2(d)$ are contained
in $N(Q(\circ))$. Since $d \circ d \circ d \in N(Q(\circ))$, $d \circ (gh(d) \circ gh(d)) \in$
$\in N(Q(\circ))$. However, $d^{-1} \circ ((d \circ d) \circ (gh(d) \circ gh(d))) =$
$= d \circ (gh(d) \circ gh(d))$ by the diassociativity of $Q(\circ)$ and

$d^{-1} \circ g^2(d)$, $d^{-1} \circ h^2(d) \in N(Q(\circ))$. Thus $d^{-1} \circ u \in N(Q(\circ))$,
and so $h^2(d^{-1} \circ u) \in N(Q(\circ))$. Multiplying (2) by $h^2(d^{-1} \circ u)$,
we obtain the equality
$(g^2(a) \circ gh(b)) \circ (hg(c) \circ h^2(u)) = (g^2(a) \circ gh(c)) \circ (hg(b) \circ h^2(u))$,
and it is not so difficult to see that $ab.c(c(dd.dd)) =$
$= ac.(b(dd.dd))$.

Lemma 5. Let Q be an LF-quasigroup (RF-quasigroup) and
$x,a,b \in Q$. Then
(i) $ef(x) = fe(x)$,
(ii) $L_b R_a = R_a L_b$ iff $e(b) = f(a)$.

Proof. (i) $x(ef(x).e(x)) = f(x)x.ef(x) e(x) =$
$= f(x).xe(x) = x = xe(x)$.

(ii) If $L_b R_a = R_a L$ then $ba = R_a L_b(e(b)) = L_b R_a(e(b)) =$
$= b.e(b)a$. Conversely, if $e(b) = f(a)$ then $b.ya = by.e(b)a =$
$= by.a$ for each $y \in Q$.

Lemma 6. Let Q be an F-quasigroup and $a,b \in Q$. Suppose
that $L_b R_a = R_a L_b$ and $R_a^{-1}(x).L_b^{-1}(y) = R_a^{-1}(y).L_b^{-1}(x)$ for all
$x,y \in Q$. Then Q is a WAD-quasigroup.

Proof. Put $x \circ y = R_a^{-1}(x).L_b^{-1}(y)$. Clearly, $Q(\circ)$ is a
commutative loop. Let $k(x) = R_a R_{f(a)} R_a^{-1}(x)$ and $t(x) =$
$= L_b L_{e(b)} L_b^{-1}(x)$ for every $x \in Q$. As it is easy to see,
$R_a(x \circ y) = R_{f(a)} R_a^{-1}(x).R_a L_b^{-1}(y) = k(x) \circ R_a(y)$ and $L_b(x \circ y) =$
$= L_b(x) \circ t(y)$ for all $x,y \in Q$. Hence $k(x \circ y) \circ R_a(j) =$
$= k(x) \circ (k(y) \circ R_a(j))$ and $L_b(j) \circ t(x \circ y) = (L_b(j) \circ t(x)) \circ t(y)$.
Now we can write
$R_a(x) \circ (L_b R_a(y) \circ t L_b(z)) = R_a(x) \circ L_b(R_a(y) \circ L_b(z)) = x.yz =$
$= xy.e(x)z = R_a(R_a(x) \circ L_b(y)) \circ L_b(R_a(e(x)) \circ L_b(z)) =$
$= (k R_a(x) \circ R_a L_b(y)) \circ (L_b R_a e(x) \circ t L_b(z))$

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for all $x,y,z \in Q$. Hence

$$x \circ (y \circ z) = (k(x) \circ y) \circ (L_b R_a e(R_a^{-1}(x)) \circ z),$$

$$(x \circ y) \circ (L_b R_a e R_a^{-1} k^{-1}(x) \circ z) = k^{-1}(x) \circ (y \circ z),$$

$$k^{-1}(x) = x \circ L_b R_a e R_a^{-1} k^{-1}(x)$$

for all $x,y,z \in Q$. According to Lemma 1, $Q(\circ)$ is a commutative Moufang loop, $L_b R_a e R_a^{-1} k^{-1}(x) = x^{-1} \circ k^{-1}(x)$ and $x^{-1} \circ x^{-1} \circ k^{-1}(x) \in N(Q(\circ))$. Therefore $x \circ k^{-1}(x) \in N(Q(\circ))$ for every $x \in Q$. Similarly we can prove that $x \circ t^{-1}(x) \in N(Q(\circ))$ for every $x \in Q$. Further, $k(x) \circ R_a(y) = R_a(x \circ y) = R_a(y \circ x) = k(y) \circ R_a(x)$, $x \circ R_a k^{-1}(y) = y \circ R_a k^{-1}(x)$ and $R_a(j) = k(j) \circ R_a(j)$. Hence $k(j) = j$, $R_a k^{-1}$ is a middle regular permutation of $Q(\circ)$ and $R_a(j) = R_a k^{-1}(j) \in N(Q(\circ))$. Similarly, $L_b(j) \in N(Q(\circ))$. Now it is obvious that both k and t are automorphisms of $Q(\circ)$ and $xy = (k(x) \circ t(y)) \circ (R_a(j) \circ L_b(j))$ for all $x,y \in Q$. Since $x^{-1} \circ k(x^{-1})$, $k(x) \circ k t^{-1}(x) \in N(Q(\circ))$ for every $x \in Q$, $k t^{-1}$ is a nuclear mapping. Finally, $tk(x) \circ t R_a(j) \circ L_b(j) = t R_a(x) \circ L_b(j) = L_b R_a(x) = R_a L_b(x) = kt(x) \circ k L_b(j) \circ R_a(j)$. Thus $t R_a(j) \circ L_b(j) = k L_b(j) \circ R_a(j)$ and $tk = kt$. An application of Theorem 1 completes the proof.

Theorem 2. The following conditions are equivalent for every quasigroup Q:

(i) Q is a WAD-quasigroup and Q is an LF-quasigroup.

(ii) Q is a WAD-quasigroup and Q is an RF-quasigroup.

(iii) Q is a WAD-quasigroup and Q is an F-quasigroup.

(iv) Q is a WAD-quasigroup and $(a.aa)(bc) = (ab)(aa.c)$ for all $a,b,c \in Q$.

(v) Q is a WAD-quasigroup and $(bc)(aa.a) = (b.aa)(ca)$ for all $a,b,c \in Q$.

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(vi) There are a commutative Moufang loop $Q(\circ)$, $g,h \in$ $\in \operatorname{Aut} Q(\circ)$ and $x \in N(Q(\circ))$ such that $gh = hg$, $a \circ g(a)$, $a \circ h(a) \in N(Q(\circ))$ and $ab = (g(a) \circ h(b)) \circ x$ for all $a,b \in Q$.

(vii) If $a,b,c,d \in Q$ and $ab.cd = ac.bd$, then the subquasigroup generated by these elements is abelian.

(viii) Q is a triabelian quasigroup.

(ix) Every subgroupoid of Q which is generated by at most three elements is abelian.

(x) Q is an F-quasigroup and there exists $z \in Q$ such that $f(z)a.be^2(z) = f(z)b.ae^2(z)$ for all $a,b \in Q$.

(xi) Q is an F-quasigroup and there exists $z \in Q$ such that $f^2(z)a.be(z) = f^2(z)b.ae(z)$ for all $a,b \in Q$.

   Proof. The implications (i) implies (vi), (ii) implies (vi), (iii) implies (vi), (iv) implies (vi) and (v) implies (vi) follow from Lemma 3 and Theorem 1.

   (vi) implies (vii). As it is easy to see, gh^{-1} is a nuclear mapping. By Theorem 1 and Lemma 4, Q is an F-quasigroup and a WAD-quasigroup. With respect to Lemma 2 and Lemma 3, we may assume that $j = dd.dd$. Then (by Lemma 4(ii)) $ab.cj = ac.bj$ and $(g^2(a) \circ gh(b)) \circ gh(c) = (g^2(a) \circ gh(c)) \circ gh(b)$. Let $G(\circ)$ be the subloop of $Q(\circ)$ generated by $N(Q(\circ)) \cup \cup \{g^2(a), gh(b), gh(c)\}$. According to the well-known Moufang theorem, $G(\circ)$ is an abelian group. Since $x \in N(Q(\circ))$ and $z \circ g(z)$, $z \circ h(z)$, $z \circ g^{-1}(z)$, $z \circ h^{-1}(z) \in N(Q(\circ))$, $g(G) = h(G) = G$, G is an abelian subquasigroup in Q and $a,b,c \in G$. Finally, $(g^2(d) \circ gh(d)) \circ (gh(d) \circ h^2(d)) = x^{-1} \circ g(x^{-1}) \circ h(x^{-1}) \in N(Q(\circ))$ and $g^2(d^{-1}) \circ d^{-1}$, $h^2(d^{-1}) \circ d^{-1}$, $gh(d^{-1}) \circ d^{-1}$, $d \circ d \circ d \in N(Q(\circ))$. Thus $d^{-1} \in N(Q(\circ))$ and $d \in G$.

(vii) implies (viii). This implication is obvious, since $ab.bc = ab.bc$. The implications (viii) implies (ix) and (ix) implies (iv), (v) are trivial and the implication (vi) implies (i), (ii), (iii) follows from Theorem 1 and Lemma 4(i).

(x) implies (i). Put $x = fe(z)$ and $y = e^2(z)$. By Lemma 5, $e(x) = f(y)$ and $R_y L_x = L_x R_y$. According to the hypothesis, $f(z)R_y^{-1}(a).b = f(z)R_y^{-1}(b).a$ for all $a,b \in Q$. Hence $R_y^{-1}(a).L_x^{-1}(b) = L_{f(z)}^{-1}(f(z)R_y^{-1}(a).b) = R_y^{-1}(b).L_x^{-1}(a)$ and we can use Lemma 6.

Similarly we can prove that (xi) implies (i).
The remaining implication (viii) implies (x),(xi) is trivial.

Corollary 1. A quasigroup Q is triabelian iff it satisfies the identity $((aa.bc)(xy.zz))((uv.wu)((r.rr)(st))) = ((ab.ac)(xz.yz))((uw.vu)((rs)(rr.t)))$ for all $a,b,c,x,y,z,u,v,w,r,s,t \in Q$.

Corollary 2. Triabelian quasigroups are finitely based.

Corollary 3. Every commutative F-quasigroup is triabelian.

Proof. Let Q be a commutative F-quasigroup and $z \in Q$. Then $f(z)a.be^2(z) = e(z)a.e^2(z)b = e(z).ab = e(z).ba = f(z)b.ae^2(z)$ for all $a,b \in Q$.

Corollary 4. Every triabelian quasigroup is isotopic to a totally symmetric triabelian quasigroup with at least one idempotent element.

Corollary 5. Every totally symmetric quasigroup isotopic to a Moufang loop is triabelian.

Proof. Let Q be a totally symmetric quasigroup isoto-

pic to a Moufang loop, $z \in Q$, $g = L_z$ and $a \circ b = g(a) \cdot g(b)$ for all $a, b \in Q$. Then $Q(\circ)$ is a commutative Moufang loop ($Q(\circ)$ is clearly commutative and every loop isotopic to a Moufang loop is Moufang) and $h(a) \circ h(h(a) \circ h(b)) = b$ for all $a, b \in Q$ and $h = g^{-1}$. For $a = g(j)$ we obtain the equality $h^2(b) = b$. Hence $h(a) \circ h^2(a) = h(a) \circ a = h(a) \circ h(h(a) \circ h(y)) = y$ for all $a \in Q$ and $y = g(j)$. Thus $h(a) = y \circ a^{-1}$ and we can write $x \circ (a^{-1} \circ (x^{-1} \circ (a \circ b))) = a \cdot ab = b$ for all $a, b \in Q$ and $x = = y \circ y$. Now it is visible that $a^{-1} \circ (x^{-1} \circ (a \circ b)) = = x^{-1} \circ (a^{-1} \circ (a \circ b))$ and $x^{-1} \in N(Q(\circ))$. Consequently $x \in \in N(Q(\circ))$ and we can use Theorem 2(vi).

Let Q be a quasigroup. A mapping g of Q into Q is called left regular if there is a mapping h such that $g(xy) = = h(x) \cdot y$ for all $x, y \in Q$.

Theorem 3. Let Q be a triabelian quasigroup. Define a binary relation r on Q by $a \, r \, b$ iff $a = t(b)$ for some left regular mapping t. Then

(i) If $(Q(\circ), g, h, x)$ is an arithmetical form of Q and a, $b \in Q$ then $a \, r \, b$ iff $b = a \circ y$ for some $y \in N(Q(\circ))$.

(ii) r is a normal congruence relation of Q.

(iii) The factorquasigroup Q/r is an idempotent totally symmetric triabelian quasigroup.

(iv) The set $\{z \mid z \, r \, a\}$ is an abelian subquasigroup in Q for every $a \in Q$.

Proof. (i) Let $a = t(b)$ for a left regular mapping t. Then there is a mapping s such that $t((g(c) \circ h(d) \circ x) = = (gs(c) \circ h(d)) \circ x$ for all $c, d \in Q$. Substituting $h^{-1}(x^{-1})$ for d we obtain the equality $tg(c) = gs(c)$. Hence $t((c \circ d) \circ x) =$

= (t(c) ∘ d) ∘ x for all c,d ∈ Q, so that t(c ∘ x) = t(c) ∘ x.
Consequently t(d) = t(j) ∘ d and t(j) ∈ N(Q(∘)). The rest
is obvious.

(ii) can be proved easily using (i).

(iii) Let (Q(∘),g,h,x) be an arithmetical form of Q
and a ∈ Q. Then (Lemma 3) a ∘ g(a), a ∘ h(a), x ∈ N(Q(∘)). Hence the elements $a^{-3} ∘ ((a ∘ g(a)) ∘ (a ∘ h(a))) =$
$= a^{-3} ∘ ((a ∘ a) ∘ (g(a) ∘ g(a))) = a^{-1} ∘ (g(a) ∘ h(a))$ and k(a) =
$= a^{-1} ∘ ((g(a) ∘ h(a)) ∘ x)$ belong to N(Q(∘)). Further,
a ∘ k(a) = aa, $a = aa ∘ (k(a))^{-1}$ and a r aa by (i). Thus Q/r
is an idempotent quasigroup. The rest is an easy consequence
of the fact that a ∘ a ∘ a ∈ N(Q(∘)) for every a ∈ Q.

(iv) follows from (i), (iii) and Lemma 2.

Corollary 6. Every simple triabelian quasigroup is abelian.

Proof. Let Q be a simple triabelian quasigroup with an
arithmetical form (Q(∘),g,h,x). Consider the normal congruence relation r defined in Theorem 3. If r = Q × Q, then Q is
abelian by Theorem 3(iv). Let r ≠ Q × Q. Then r is the identical relation (since Q is simple) and Q is idempotent and totally symmetric as it follows from Theorem 3(iii). In this
case, $g(a) = h(a) = a^{-1}$ for every a and x = j. It is easy to
see that every congruence of Q(∘) is a congruence of Q, and
consequently Q(∘) is simple. However, every simple commutative Moufang loop is a group.

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