COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,1 (1976)

A NOTE ON STRONG DIFFERENTIABILITY SPACES

K. JOHN and V. ZIZLER, Praha

Abstract: It is shown that if X^* is weakly compactly generated (WCG) Banach space or if X is WCG and for any separable subspace $Y \subset X$, Y^* is separable, then any convex function on X is Fréchet differentiable on a G_δ dense subset of its domain of continuity, i.e. X is a strong differentiability space (SDS).

Key words: Convex functions, differentiability, Asplund spaces.

AMS: 46B99 Ref. Ž.: 7.972.22

A Banach space X is weakly compactly generated (WCG) if $X = \overline{sp}K$ - the closed linear hull of some weakly compact set $K \subset X$. Given a convex function f on a Banach space X, we say that f is continuous at a point $a \in X$, if f is finite valued and continuous at a. Similarly for the case of differentiability. X is called a strong differentiability space (SDS) if any convex function f on X which is (i) continuous at some point of X and (ii) with values in $(-\infty, +\infty\rangle$, is Fréchet differentiable on a G_δ dense subset G of the domain of continuity C of f ($C = \mathrm{Int\,dom} f$, where $\mathrm{dom} f = \{x \in X ; f(x) < \infty\}$). Any convex function f on X with (i) and (ii) will be called, following [1], a continuous convex function on X.

R denotes the reals and for a function f on X, $\operatorname{epi} f = \{(x,r) \in X \times R,\ f(x) \le r\}$. The Fenchel dual function f^* on X^* is $f^*(x^*) = \sup_{x \in X} (x^*(x) - f(x))$. A set $K \subset X^*$ is called w^* dentable if for any $\varepsilon > 0$, there is a $p \in K$ such that $p \notin w^* \operatorname{clconv}(K \setminus B_\varepsilon(p))$, where w^* clconv means the w^*-closed convex hull-operation and $B_\varepsilon(p)$ is the closed ε-ball centered at p. A point $p \in K \subset X$ is a w^* strongly exposed point of K if there is an $x \in X$ such that $p(x) \ge f(x)$ for each $f \in K$, and whenever $\lim((f_n - p)(x)) = 0$, $f_n \in K$, then $\lim \| f_n - p \| = 0$. WCG spaces were introduced by D. Amir, H. Corson and J. Lindenstrauss (see [11]), SDS spaces by E. Asplund ([1]). SDS were originally known to include e.g. the spaces with separable dual ([1]), then e.g. WCG spaces with Fréchet smooth norm ([17] and [7]), spaces X, for which X, X^* are both subspaces of WCG spaces ([8]). It is known that SDS property of X implies X^* to have the Radon-Nikodym property ([2],[19],[15]). Proving our results, we answer the question of R.R. Phelps in [14], p. 86.

In the sequel, the following will be suitable.

<u>Definition</u>. We say that a convex function f on X possesses the property P if it is bounded on some ball $B_\varepsilon(0)$, $f | B_\varepsilon(0)$ continuous and equals $+\infty$ outside $B_\varepsilon(0)$.

<u>Lemma 1.</u> A Banach space X is SDS provided each convex function f on X with the property P is Fréchet differentiable at some point of X.

<u>Proof.</u> Let f be a continuous convex function on X, $x \in \operatorname{Int} \operatorname{dom} f$ and $\varepsilon > 0$ is so that $|f(z)| \le K$ for $z \in B_\varepsilon(x) \subset \operatorname{Int} \operatorname{dom} f$. We will first show that there is a point

$y \in \operatorname{Int} B_\varepsilon(x)$ at which f is Fréchet differentiable. For it let us define a function g on X by

$$g(z) = f(z + x) \quad \text{for} \quad z \in B_\varepsilon(0) ,$$
$$g(z) = +\infty \quad \text{for} \quad z \notin B_\varepsilon(0) .$$

Then g has the property P and thus is, by our assumption, Fréchet differentiable at some point $y \in \operatorname{Int} B_\varepsilon(0)$. Clearly, $y + x$ is a point of Fréchet differentiability of f, $y + x \in \operatorname{Int} B_\varepsilon(x)$. So, the set G of all points of Fréchet differentiability of f is dense in Intdom f. Therefore, by Lemma 6 of [1], p. 43, G is a dense G_δ set in Intdom$f$.

<u>Lemma 2</u>. Let f be a convex function on X with the property P. Then epi f^* is w^* locally compact in $X^* \times R = (X \times R)^*$ and contains no line.

<u>Proof</u>. Let $|f(z)| \leq K$ for $z \in B_\varepsilon(0)$, $f \mid B_\varepsilon(0)$ continuous, $f = +\infty$ outside $B_\varepsilon(0)$. We first observe that f^* is finite on all of X^*: $|f^*(x^*)| = |\sup_{z \in B_\varepsilon(0)} (x^*(z) - f(z))| \leq \sup_{z \in B_\varepsilon(0)} |x^*(z)| + \sup_{z \in B_\varepsilon(0)} |f(z)| < \infty$. Furthermore, we show that for any $L > 0$, the set epi $f^* \cap X^* \times (-\infty, L\rangle$ is w^* locally compact. Clearly, epi $f^* \cap (X^* \times (-\infty, L\rangle) = (\{x^* ; f^*(x^*) \leq L\} \times (-\infty, L\rangle) \cap$ epi f^*. So, it suffices to show that the sets $M_L = \{x^* \in X ; f^*(x^*) \leq L\}$ are bounded for each $L > 0$.

For it observe that if $x^* \in M_L$, then $\sup_{z \in B_\varepsilon(0)} x^*(z) - K \leq f^*(x^*) \leq L$, so $\| x^* \| \leq \varepsilon^{-1} . (K + L)$.

Since f is finite and M_L are bounded, epi f^* is easily seen to contain no lines.

<u>Lemma 3.</u> Assume X is a Banach space such that any

w^* compact convex $K \subset X^*$ contains w^* strongly exposed points. Then for each convex function f on X with the property P , epi f^* has w^* strongly exposed points.

Proof. Easily, then any w^* compact convex $K \subset (X^* \times R)$ has w^* strongly exposed points. So, Lemma 3 follows from Th. 4 of [12] .

Proposition. If X satisfies the assumption of Lemma 3, then X is SDS.

Proof. If f is a convex function on X with the property P and $(b, f^*(b))$ is a w^* strongly exposed point of epi f^* (Lemma 3), exposed by some $(a,-1) \in X^* \times R$, then it follows that $f = (f^*)_*$ is Fréchet differentiable at $a \in \operatorname{Int} B_\varepsilon(0)$, by Prop. 1 of [1]. Further use Lemma 1.

Theorem 1. If X^* is WCG, then X is SDS.

Proof. If X^* is WCG, then the assumption of Lemma 3 is satisfied for X by [13] and [14] Cor. 11 . Furthermore we will need the following lemma, which is a variant of the result of H. Maynard ([12], p. 494).

Lemma 4. Assume X is WCG and $K \subset X^*$ is a bounded subset of X . Then K is w^* dentable if any countable $C \subset K$ is such.

Proof. Suppose K is not w^* dentable. Then there is an $\varepsilon > 0$ such that for each $x \in K$, $x \in w^*$ clconv $(K \setminus B_\varepsilon(x)) = w^*$ sequential clconv of $(K \setminus B_\varepsilon(x))$ - since X is WCG ([11], Th.3.3). So, there is for each $x \in K$ a countable set $K_x \subset K \setminus B_\varepsilon(x)$ with $x \in$ wclconvK_x . So we can define by induction a sequence K_n of subsets of K as follows.

Pick an arbitrary $z \in K$ and set $K_1 = \{z\}$. Given K_{n-1}, put $K_n = \bigcup \{K_x ; x \in K_{n-1}\}$. Then the set $\bigcup_n K_n$ is clearly a countable subset of K which is not w^* dentable.

Lemma 5. If X is WCG and for each separable subspace $Y \subset X$, Y^* is separable, then any bounded closed convex subset of X^* has w^* strongly exposed points.

Proof. By the results of [14] (see p. 86) and Lemma 4, it suffices to show that any bounded countable $C \subset X^*$ is w^* dentable. Consider $C' = \overline{sp}\, C$, s_n dense in C' and for each n, $t_j^n \in X$, $\| t_j^n \| = 1$, such that $s_n(t_j^n) \geq \| s_n \| - 1/j$. Put $T = \overline{sp} \bigcup_{n,j} t_j^n$. Then T is a separable subspace of X and if $R: X^* \longrightarrow T^*$ is the restriction map, then R is an isometry on C' and an isomorphism with respect to the topology of pointwise convergence on T. RC is w^* dentable as a bounded subset of separable T^* (see [13]), since w^* clconv RC has w^* strongly exposed points ([1], Prop. 5) and thus there is a nonempty intersection of RC with a w^* open halfspace of arbitrarily small diameter. Thus RC is w^* dentable in T^* and thus C is a fortiori w^* dentable in X^*.

Now we can summarize some known results with the above ones in the following statement, where another question of [14] (p. 86) is answered for WCG spaces.

Theorem 2. Let X be a WCG Banach space. Then the following conditions are equivalent:

(i) X is SDS,

(ii) X^* has the Radon-Nikodym property,

(iii) for each separable subspace $Y \subset X$, Y^* is separable,

(iv) any bounded subset of X^* is w^* dentable.

<u>Proof</u>. (i)$\Rightarrow$(ii) - see [2],[19],[15].

(ii)$\Rightarrow$(iii) - see [16].

(iii)$\Rightarrow$(iv) - see Lemma 5.

(iv)$\Rightarrow$(i):(iv) implies by [14], p.86 that the assumptions of Lemma 3 are fulfilled and (i) then follows by Proposition.

We may state the following

Questions. (i) Is the conclusion of Theorem 1 true if X^* is a subspace of WCG?

(ii) Is the conclusion of Theorem 2 true without the assumption on WCG of X ?

We finish with two remarks.

<u>Remark 1.</u> It is known that the separable Banach space J_0 constructed by R.C. James ([5]) has the property that all its even duals are WCG while its odd duals are not WCG (see [6], p. 220). Since J_0 is separable and J_0^* is not separable, J_0^* does not have the Radon-Nikodym property by the Stegall's theorem ([16]) and so, J_0 is not SDS (see the Introduction). Nevertheless, J_0^* is SDS, since J_0^{**} is WCG (use Theorem 1).

<u>Remark 2</u>. After the paper had been prepared for publication, the authors were learned that the results in it were also proved by J.B. Colier, P.D. Morris, I. Namioka and R.R. Phelps (to appear in Proc. Amer. Math. Soc. and Duke J. Math.).

R e f e r e n c e s

[1] E. ASPLUND: Fréchet differentiability of convex functions, Acta Math. 121(1968), 3-47.

[2] E. ASPLUND: Boundedly Krejn-compact Banach spaces, Proc. of Functional Analysis week, Aarhus (1969),

[3] E. ASPLUND, R.T. ROCKAFELLAR: Gradients of convex functions, Trans. Amer. Math. Soc. 139(1969), 443-467.

[4] J. DIESTEL, J.J. UHL: The Radon-Nikodym property for Banach spaces - valued measures, to appear.

[5] R.C. JAMES: A separable somewhat reflexive Banach space with nonseparable dual, Bull. Amer. Math. Soc. 80(1974), 738-743.

[6] W.B. JOHNSON; J. LINDEBSTRAUSS: Some remarks on weakly compactly generated Banach spaces, Israel J. Math. 17(1974), 219-230.

[7] K. JOHN, V. ZIZLER: A note on renorming of dual space, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21(1973), 47-50.

[8] K. JOHN, V. ZIZLER: On the heredity of weak compact generating, Israel J. Math. 20(1975), 228-236.

[9] V.I. KADEC: Some conditions of the differentiability of the norm of Banach spaces, Uspechi Mat. Nauk SSSR 20(1965), 183-187 (Russian).

[10] V. KLAS: Extremal structure of convex sets II, Math. Zeitschr. 69(1958), 90-104.

[11] J. LINDENSTRAUSS: Weakly compact sets, their topological properties and the Banach spaces they generate, Symp. on Infinite Dimensional Topology, Annals of Math. Studies 69,Princeton Univ. Press, (1972), 235-273.

[12] H. MAYNARD: A geometrical characterization of Banach spaces with the Radon-Nikodym property, Trans. Amer. Math. Soc. 185(1973), 493-500.

[13] I. NAMIOKA: Separate continuity and joint continuity, Pac. J. Math. 51(1974), 515-531.

[14] R.R. PHELPS: Dentability and extreme points in Banach spaces, J. Functional Analysis, 17(1974),78-90.

[15] M.A. RIEFFEL: Dentable subsets of Banach spaces with applications to a Radon-Nikodym theorem, in Functional Analysis (B.R. Gelbaum, editor).Washington: Thompson Book Co., 1967.

[16] C. STEGALL: The Radon-Nikodym property in conjugate Banach spaces, to appear.

[17] S. TROJANSKI: On equivalent norms and minimal systems in nonseparable Banach spaces, Studia Math. 43(1972), 125-138.

[18] V. ZIZLER: On extremal structure of weakly locally compact convex sets in Banach spaces, Comment.Math. Univ. Carolinae 13(1972), 53-61.

[19] V. ZIZLER: Remark on extremal structure of convex sets in Banach spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 6(1971), 451-455.

Matematický ústav ČSAV
Žitná 25
Praha 1
Československo

Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 4.8. 1975)