COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,1 (1976)

EXACTNESS OF THE SET-VALUED COLIM

J. ADÁMEK, J. REITERMAN, PRAHA

Abstract: It is well-known that, in the category of sets, filtered colimits commute with finite limits; thus, if K is a filtered small category then the functor colim: $Set^K \longrightarrow Set$ is exact (i.e. preserves regular epis and finite limits). The converse is proved in the present note and other properties of colim are investigated and compared with these of colim: $Ab^K \longrightarrow Ab$ for the category Ab of Abelian groups.

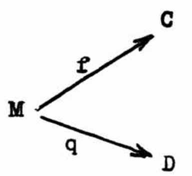
Key words: Exact colimits, category of sets.

AMS: 08A10, 18B05 Ref. Ž.: 2.726

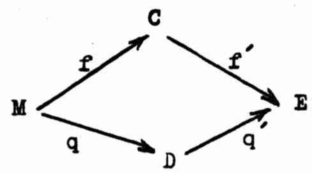
I. Formulation

I.1. The exactness of colim for Ab has been investigated by Isbell and Mitchell [2], [3]. In that case colim is exact iff it preserves equalizers and iff it preserves monics. For the set-valued colim (i.e. for colim : $Set^K \longrightarrow Set$) these properties differ. We shall prove namely the following propositions (see part III).

I.2. (a) colim preserves monics iff every diagram (*)



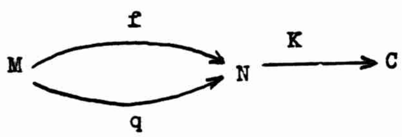
(*)



(**)

in K is a part of commutative square (**)

(b) colim preserves equalizers iff K has filtered components, i.e. iff K fulfils the condition of (a) and for every pair f, g of parallel morphisms there is k with kf = kg ,



(c) colim is exact iff K is filtered, i.e. iff K fulfils the conditions of (a),(b) and for every pair A, B of K-objects there is C with $Hom(A,C) \neq \emptyset \neq Hom(B,C)$.

I.3. This characterization is rather simple in comparison with the Ab case. Colim: $Ab^K \longrightarrow Ab$ is exact iff the following category aff K has filtered components: objects of aff K are just the objects of K ; morphisms from A to B are those elements $\sum \alpha_i f_i$ of the free Abelian group over $Hom_K(A,B)$ for which $\sum \alpha_i = 1$, see [3] .

I.4. It is easily seen that 1) aff K has filtered components provided that K has, 2) if aff K has filtered components then K fulfils the condition of (a) . Thus, denoting $A = \text{colim} : Ab^K \longrightarrow Ab$, $S = \text{colim} : Set^K \longrightarrow Set$ we get

S is exact $\Longrightarrow$ S preserves equalizers $\Longrightarrow$ A is exact $\Longrightarrow$ S preserves monics

None of these implications can be reversed. The counterexamples are easy (according to I.2, I.3) except that to the second implication: for the category K of finite ordinals and order preserving injections, A is proved to be exact in [3] but the only component of K is not filtered.

II. Relation to indecomposable functors

II.1. Colimits in sets are closely related to indecomposability: a functor $F: K \longrightarrow Set$ is indecomposable if whenever $F = F_1 \vee F_2$ then F_1 or F_2 is the constant functor to $\emptyset$. Notice that F is indecomposable iff colim F is a singleton set.

Let us observe that each non-trivial functor $F: K \longrightarrow Set$ can be decomposed into a sum of its components, i.e. maximal indecomposable subfunctors, $F = \coprod_{i \in I} F_i$. If $\mu : F \longrightarrow F'$ is a transformation and $F' = \coprod_{j \in J} F'_j$ is a decomposition of F' into components then for every $i \in I$ there is $c(i) \in J$ with $\mu(F_i) \subset F_{c(i)}$. We have colim $F = I$, colim $F' = J$, colim $\mu = c$. From these observations one can derive the following properties of colim: $Set^K \longrightarrow Set$.

II.2. (a) colim preserves monics iff each non-trivial subfunctor of an indecomposable functor $F: K \longrightarrow Set$ is indecomposable, too.

(b) colim preserves equalizers iff indecomposable functors from K to Set have always the following "agreement property": for each couple $\mu, \nu : F \longrightarrow F'$ of transformations there is M and $x \in FM$ with $\mu_M x = \nu_M x$.

(c) colim preserves finite products iff the product of two indecomposable functors from K to Set is indecomposable, too.

II.3. The exactness of colim in the Ab case can be also characterized analogously [1] : colim: $Ab^K \longrightarrow Ab$ is exact iff the agreement property from (b) holds for all couples of endo-transformations of indecomposable functors from K to Set ; equivalently, iff each endotransformation μ : $: F \longrightarrow F$ of an indecomposable functor $F: K \longrightarrow Set$ has a fixed point (i.e. x in some FM with $\mu_M x = x$).

III. <u>Proof</u>

III.1. <u>Necessities</u> in I.2 follow from II.2 if we take into account that

(a) the subfunctor F of Hom(M,-) generated by f: $: M \longrightarrow C$, $g: M \longrightarrow D$ must be indecomposable (then we have $f': C \longrightarrow E$, $g': D \longrightarrow E$ with $f'f = g'g$),

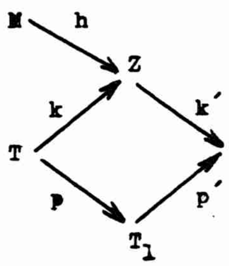
(b) the transformations Hom(f,-) , Hom(g,-): : Hom(N,-) $\longrightarrow$ Hom(M,-) must coincide at some $k \in$ Hom(N,C) ; and all monics are equalizers in Set^K ,

(c) the product Hom(M,-)$\times$ Hom(N,-) must be non-trivial.

III.2. <u>Sufficiencies</u>. (a) Let $F: K \longrightarrow Set$ be an indecomposable functor. To prove that all subfunctors of F

are indecomposable it suffices, for given $x \in \mathcal{F}M$, $y \in \mathcal{F}N$, to find $h: M \to Z$, $k: N \to Z$ with $\mathcal{F}h(x) = \mathcal{F}k(y)$. Fix $x \in \mathcal{F}M$.

For every object T put $HT = \{ t \in \mathcal{F}T$; there are $h:$ $: M \to Z$, $k: T \to Z$ with $\mathcal{F}h(x) = \mathcal{F}k(t) \}$; we shall prove that $H = \mathcal{F}$. First, H is a subfunctor of $\mathcal{F}$: given $t \in HT$ and given a morphism $p: T \to T_1$ we have $h: M \to Z$, $k:$ $: T \to Z$ with $\mathcal{F}h(x) = \mathcal{F}k(t)$; since p, k have a common domain there exist p', k' with $p'p = k'k$. This proves $\mathcal{F}p(t) \in HT_1$, because $\mathcal{F}(k'k)(x) = \mathcal{F}p'(\mathcal{F}p(t))$.



Second, $\mathcal{F} - H$ (defined by $(\mathcal{F} - H)T = \mathcal{F}T - HT$) is a subfunctor of $\mathcal{F}$, as is easily seen. Since $\mathcal{F}$ is indecomposable and $\mathcal{F} = H \vee (\mathcal{F} - H)$, either $\mathcal{F} = H$ or $\mathcal{F} = \mathcal{F} - H$. The latter cannot occur, since $x \in HM$.

(b) Let $\mu, \nu : \mathcal{F} \to \mathcal{F}'$ be transformations between non-trivial indecomposable functors. Choose $z \in \mathcal{F}M$ arbitrarily and put $x = \mu_M z$, $y = \nu_M z$. Via the previous part of the proof there exist $h, k: M \to Z$ with $\mathcal{F}'h(x) = \mathcal{F}'k(y)$ Choose $p: Z \to T$ with $ph = pk$ and put $t = \mathcal{F}(ph)(x)$. Then $\mu_T t = \mathcal{F}'(ph)(z) = \mathcal{F}'(pk)(z) = \nu_T t$.

(c) is well known.

This concludes the proof.

IV. A corollary

IV.1. Let T be a cocomplete category which has a full subcategory D isomorphic to Set and closed under colimits and finite limits. Then we have

colim: $T^K \longrightarrow T$ is exact $\Longrightarrow$ K is filtered.

Indeed, if colim: $T^K \longrightarrow T$ is exact so is colim: : $D^K \longrightarrow D$, the latter being a restriction of the former one. As $D \sim$ Set , K is filtered by I.2c.

IV.2. The above corollary applies e.g. to the category of

- topological (resp. uniform) spaces,
- graphs,
- unary algebras of a given type

and to T^L for any such T and any small L .

In all of these examples filtered colimits commute with finite limits (as is easily seen) so that we have

colim: $T^K \longrightarrow T$ is exact $\Longleftrightarrow$ K is filtered.

References

[1] J. ADÁMEK, J. REITERMAN: Fixed points in representations of categories, Trans. Amer. Math. Soc. 211(1975), 239-247.

[2] J.R. ISBELL: A note on exact colimits, Canad. Math. Bull. 11(1968), 569-572.

[3] J.R. ISBELL and B. MITCHELL: Exact colimits, Bull. Amer. Math. Soc. 79(1973), 994-996.

Elektrotechnická fakulta
ČVUT
Suchbátarova 2,16627 Praha 6
Československo

Fakulta jaderná a fyzikálně
inženýrská ČVUT
Husova 5, 11900 Praha 1
Československo

(Oblatum 2.6. 1975)