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PARTITIONS OF VERTICES

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Abstract: In this note we show how a theorem by Erdős-Hajnal may be used for proving theorems concerned with partitions of vertices of graphs, relations etc.

Key-words: Ramsey theorem, partitions, chromatic number.

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Introduction. In 1966 Erdős and Hajnal [1] proved the following.

Theorem A : For every positive integer $k \geq 2$, $\ell \geq 2$, $n \geq 1$ there exists a hypergraph $\mathcal{G} = \mathcal{G}(k, \ell, n) = (X, \mathcal{M})$ with the following properties:

1) $\mathcal{G}$ is a k-uniform hypergraph

2) $\mathcal{G}$ does not contain cycles of length smaller than ℓ

3) $\chi(\mathcal{G}) > n$

The notation is the following: $\chi(\mathcal{G})$ = chromatic number of $\mathcal{G}$ i.e. the minimal number of colours which are necessary for colouring the vertices of $\mathcal{G}$ in such a way that no monocoloured hyperedge occurs; k-uniform means that $|M| = k$ for every $M \in \mathcal{M}$; a cycle of length ℓ is a sequence $x_1, M_1, x_2, M_2, \dots, x_\ell, M_\ell$ such that $x_i \in M_i$, $i \in [1, \ell]$;

$x_{i+1} \in M_i$, $i \in [1, \ell - 1]$; $x_1 \in M_\ell$

$\{M_i \mid i \in [1, \ell]\} \subseteq \mathcal{M}$, $\{x_i \mid i \in [1, \ell]\} \subseteq X$.

To avoid the trivial cycle consisting of only one hyperedge we assume that there are i, j such that $M_i \neq M_j$.

Theorem A was proved by nonconstructive means. In 1968 L. Lovász proved the same theorem constructively.

In this note we show how this theorem implies (using a simple trick) a very general theorem of Ramsey type for partitions of vertices. There are two reasons for publishing of this note: first, the trick provides simpler proofs to known theorems ([2],[3],[4]), secondly, partitions of vertices are used as a tool for proving a Ramsey type theorem for partitions of edges and we shall need a general theorem for partitions of vertices for our forthcoming papers.

We apply the Theorem A to partitions of vertices of graphs, hypergraphs, relations and universal algebras. In § 4 we show that given a graph G there exists an infinite set of minimal graphs with the vertext partition property for G . We end this note with a few problems and comments concerning infinite graphs.

1. Folkman's theorem. In 1967 J. Folkman [3] proved: For every positive integer r and for every graph $G = (V,E)$ without complete subgraphs on m vertices there exists a graph $H = (W,F)$ without complete subgraphs on m vertices such that for every partition $W = \bigcup_{i=1}^{r} W_i$ there exists an i and an embedding $f: G \longrightarrow H$ such that $f(V) \subseteq W_i$ (An embedding $f: G \longrightarrow H$ is an 1-1 mapping with the property

$\{f(x),f(y)\} \in F \Longleftrightarrow \{x,y\} \in E$.). We denote by $G \xrightarrow[\mathcal{K}]{v} H$ the validity of the above statement for G, H , the negation is denoted by $G \not\xrightarrow[\mathcal{K}]{v} H$. This notation has the following sense: Let $G \longrightarrow H$ denote the fact that there exists an embedding of G into H . Then $G \xrightarrow[\mathcal{K}]{v} H$ means that there are "so many"embeddings of G into H that even if we partition vertices of H into r parts we still have an embedding in one of the parts. In this way $\xrightarrow[\mathcal{K}]{v}$ may be seen as a combinatorially strengthened embedding arrow (see [8]) .

Folkman gave a direct constructive proof of the above fact. An another (less elementary) proof is due to the authors of [7]. However, Theorem A instantly yields a much stronger result.

<u>Definition</u>: Let K be a fixed graph. Denote by Gra (K) the class of all graphs which do not contain K as a subgraph. (I.e. $G \notin$ Gra (K) $\Longleftrightarrow$ there are sets $V_0 \subseteq V$, $E_0 \subseteq E$ such that $(V_0,E_0) \cong K$.)

If $\mathcal{K}$ is a set of graphs put Gra $(\mathcal{K}) = \bigcap$ (Gra (K) $|$ $K \in \mathcal{K}$).

<u>Theorem 1:</u> Let $\mathcal{K}$ be a finite set of 2-connected graphs. Then for every graph $G \in$ Gra $(\mathcal{K})$ there exists a graph $H \in$ Gra $(\mathcal{K})$ such that $G \xrightarrow[\mathcal{K}]{v} H$.

We may assume $|K| > 2$ for every $K \in \mathcal{K}$ as for $|K| \leq 2$ we get either the void class of graphs of the class of all discrete graphs.

<u>Proof</u>: Let $G = (V,E)$ Gra $(\mathcal{K})$ be fixed. Let $b = \max_{K \in \mathcal{K}} |K| + 1$, $|G| = k$. Let us choose $\mathcal{S}(k,\ell,r) = (X,\mathcal{M})$

with the properties of Theorem A. For each $M \in \mathcal{M}$ let $f_M : V \longrightarrow M$ be a fixed bijection. Define the graph $H = (X,F)$ such that $\{x,y\} \in F \Longleftrightarrow$ there exist $M \in \mathcal{M}$ and $\{z,t\} \in E$ such that $\{f_M(z), f_M(t)\} = \{x,y\}$. This graph H will be denoted by $(X,\mathcal{M}) * G$.

As $\ell > 2$ we have $|M \cap N| \leq 1$ whenever $M \neq N$, $\{M,N\} \subseteq \mathcal{M}$ (see above) and consequently

1) $f_M : V \longrightarrow M$ is an embedding of G into H for each $M \in \mathcal{M}$.

2) If K' is a subgraph of H, $K' \cong K \in \mathcal{K}$ then $K' \subseteq (M,F)$, $M \in \mathcal{M}$.

(This follows by the 2-connectivity of K and by the fact that $(X,\mathcal{M})$ does not contain a cycle of length $< |K| + 1$.)

Finally $G \xrightarrow[r]{v} H$ follows immediately from $\chi(X,\mathcal{M}) > r$:

Given a partition $X = \bigcup_{i=1}^{r} X_i$ there exists $M \in \mathcal{M}$ and $i \in [1,r]$ such that $M \subseteq X_i$. Consequently G is an induced subgraph of (X_i,F) and f_M is an embedding.

This theorem does not hold for graphs with connectivity < 2:

i) If K is disconnected and $K = K' \cup K''$ where $K' \cong K''$ then $H \not\xrightarrow[r]{v} K'$ for every $H \in \mathrm{Gra}(K)$ as may be seen easily.

ii) If $K = P_n$ is a path of length n then $G \in \mathrm{Gra}(P_n) \Longrightarrow \chi(G) \leq n$. From this follows that there exists $G \in \mathrm{Gra}(P_n)$ such that $G \not\xrightarrow[r]{v} H$ for every graph $H \in \mathrm{Gra}(P_n)$ (it suffices to take $G \in \mathrm{Gra}(P_n)$ which satisfies $\chi(G) > \frac{n}{2}$; obviously $G \xrightarrow[2]{v} H \Longrightarrow \chi(H) \geq$

$\geq 2\ \chi(G) - 1$) .

iii) If $\mathcal{K}$ is an infinite set then the statement may be false (consider $\mathcal{K} = \{C_{2k+1} \mid k \leq 1\}$ the set of all odd cycles).

2. Partitions of vertices of relations and hypergraphs. Using the same ideas as in 1 we may prove analogous theorems for relations and hypergraphs. We list only statements:

Theorem 2a : Let $\mathcal{R}$ be a finite set of 2-weakly connected relations (see [5] , p.199) . Then for every positive integer r and for every $R \in \mathcal{R}el(\mathcal{R})$ there exists $S \in \mathcal{R}el(\mathcal{R})$ such that $R \xrightarrow[\mathcal{r}]{v} S$.

Theorem 2b: Let $\mathcal{S}$ be a finite of hypergraphs which are 2-connected (i.e. $(X,\mathcal{M})$ is 2-connected $\Longleftrightarrow$ $\Longleftrightarrow (X, \bigcup (P_r(M) \mid M \in \mathcal{M}))$ is a 2-connected graph). Then for positive integer r and for every $(X,\mathcal{M}) \in Hyp(\mathcal{S})$ there exists $(Y,\mathcal{N}) \in Hyp(\mathcal{S})$ such that

$(X,\mathcal{M}) \xrightarrow[\mathcal{r}]{v} (Y,\mathcal{N})$.

The definitions of $\mathcal{R}el(\mathcal{R})$ and $Hyp(\mathcal{S})$ and of symbols $\xrightarrow[\mathcal{r}]{v}$ are quite analogous to the definitions $Gra(\mathcal{K})$ and $\xrightarrow[\mathcal{r}]{v}$ for graphs. Again in certain sense these theorems are best possible.

3. Partitions of algebras. Let $\mathcal{V}$ be the class of all finite universal algebras of given type $\Delta = (n_i \mid i \in I)$.

Let $\mathfrak{X} = (X,(\omega_i \mid i \in I))$, $\mathcal{Y} = (Y,(\varkappa_i \mid i \in I))$ be algebras from $\mathcal{V}$. We write $\mathfrak{X} \xrightarrow[r]{} \mathcal{Y}$ iff for every partition $Y = \bigcup_{i=1}^{r} Y_i$ there exists an i and a monomorphism f: : $\mathfrak{X} \longrightarrow \mathcal{Y}$ such that $f(x) \subseteq Y_i$.

Theorem 3: Let r be positive integer and let $\mathfrak{X} \in \mathcal{V}$ and $\omega_i(x,x,\dots x) = x$ for every $i \in I$ and $x \in X$ (idempotent algebras). Then there exists $\mathcal{Y} \in \mathcal{V}$ such that $\mathfrak{X} \xrightarrow[r]{} \mathcal{Y}$.

Proof: Let $|X| = k$, $\ell = 3$, $n = r$. Consider $(Y',\mathcal{M}) = \mathcal{S}(k,3,r)$. Let $y' \in Y'$. For every $M \in \mathcal{M}$ let us choose a bijection $f_M: X \longrightarrow M$. Define $(Y,(\varkappa_i \mid i \in I))$ by $Y = Y' \cup \{y'\}$ and $\varkappa_i(y_j) \mid j \in [1,n_i]) =$ $= f_M(\omega_i(x_j \mid j \in [1,n_i]))$ where $f_M(x_j) = y_j$ if such an M exists, otherwise we put $\varkappa_i(y_j \mid j \in [1,n_i]) = y'$, $i \in I$. It is easy to check that $\mathcal{Y} \in \mathcal{V}$, $\mathfrak{X} \xrightarrow[r]{} \mathcal{Y}$.

Again it is easy to see that, generally, for non-idempotent algebras Theorem 3 fails to be true.

Remark: A very difficult problem seems to be the characterization of those primitive classes of algebras for which the statement analogous to Theorem 3 holds. This is true for example for the class of all finite distributive lattices.

4. Critical Folkman graphs: Let G be a graph. We say that H is an irreducible (r,v)-graph for G if $G \xrightarrow[r]{v} H$ but $G \not\xrightarrow[r]{v} H'$ for every proper subgraph

H' of H.

<u>Theorem 4 a</u>: For every graph G, $|G|>1$ there exists a countable set of non-isomorphic irreducible (r,v)-graphs for G.

<u>Proof</u>: A proof follows directly from the $*$ construction in 1. Let G be fixed. We may assume that G is a connected graph (otherwise we consider the complement of G).

It suffices to put

$$H_1 = \mathcal{G}(|G|, 3, r) * G$$
$$H_2 = \mathcal{G}(|G|, |H_1|, r) * G$$
$$\vdots$$
$$H_{n+1} = \mathcal{G}(|G|, |H_n|, r) * G$$
$$\vdots$$

$G \xrightarrow[r]{v} H_i$ holds for every i. Let $\overline{H}_i$ be an irreducible (r,v)-graph for G contained in H_i, $i = 1,2,\ldots$. Obviously $|H_i| < |H_j|$ for all i, j satisfying $i<j$. Assume $\overline{H}_i \cong \overline{H}_j$ for $i<j$. As $\overline{H}_j \subseteq H_j$, $H_j =$ $= \mathcal{G}(|G|, |H_{j-1}|, r) *$, and $|H_{j-1}| \geq |H_i|$ we have $|\overline{H}_i| \subseteq \overline{\mathcal{G}} * G$ where $\overline{\mathcal{G}} \subseteq \mathcal{G}(|G|, |H_{j-1}|, r)$ is a hypergraph which does not contain any cycle. But in this case $\chi(\overline{\mathcal{G}} * G) = \chi(G)$. This contradicts $G \xrightarrow[r]{v} \overline{H}_i$.

<u>Remark 1:</u> Using a modified proof we may even prove

Theorem 4 b: Let $\mathcal{K}$ be a finite set of 2-connected graphs. Then for every graph $G \in Gra(\mathcal{K})$ there exists a countable set $\{H_i \; i = 1,2,\dots\}$ of non-isomorphic graphs such that

1) $H_i \in Gra(\mathcal{K})$
2) H_i is an irreducible (r,v)-graph for G.

Remark 2: Theorem 4a does not hold for infinite graphs. Every complete graph of infinite cardinality is the only (r,v)-irreducible graph for itself. Theorem 4a fails to be true for $|G| = 1$, too.

Remark 3: Let $G = (V,E)$, $H = (W,F)$ be graphs. We write $G \xrightarrow[r]{e} H$ if for every partition $F = \bigcup_{i=1}^{r} F_i$ there exists an embedding $f: G \to H$ such that $\{\{f(x),f(y)\} \mid \{x,y\} \in E\} \subseteq F_i$ for an $i \in [1,r]$. The existence of an Ramsey graph for every finite graph was proved independently by Deuber, Erdős, Hajnal, Posa and Rödl [9], see also much stronger [7].

Define H to be an (r,e)-irreducible graph for G if $G \xrightarrow[r]{e} H$ but $G \not\xrightarrow[r]{e} H'$ for all proper subgraphs H' of H.

Problem 1: Characterize those finite graphs G for which there exists an infinite set of non-isomorphic (r,e)-irreducible graphs for G.

If a graph G contains at most one edge then there exists precisely one (r,e)-irreducible graph H such that $G \xrightarrow[r]{e} H$, namely G itself.

Conjecture: For a finite graph G the following two statements are equivalent:

1. For G there exists a countable set of (r, e)-irreducible graphs.

2. $G \not\xrightarrow[r]{e} G$.

The path of length 2 is an example of a graph G for which there exists a countable set of $(2, e)$-irreducible graphs. One can take the family of all odd cycles.

More generally, the same is true for every paths of length ℓ, ℓ finite.

Finally let us remark that Theorem 1 shows the power of Erdős-Hajnal theorem for partitions of vertices.

There is no general method known for deriving similar theorems for partitions of edges (see [8] for results in this direction). Let us add a few remarks concerning infinite graphs. In an obvious way we may extend the symbol $G \xrightarrow[r]{v} H$ for infinite graphs G, H and any cardinal r. The following is then true:

Theorem 5a: For every graph G and every positive integer r there exists H such that $G \xrightarrow[r]{v} H$.

Theorem 5b: For every finite graph G and every cardinal r there exists a graph H such that $G \xrightarrow[r]{v} H$. Moreover, if G does not contain a complete graph on m vertices then H may be chosen with the same property.

Theorem 5a may be proved by the following construction: Let $G = (V,E)$, assume without loss of generality $r = 2$

(this is possible as $G \xrightarrow[2]{v} H \xrightarrow[2]{v} I \Longrightarrow G \xrightarrow[4]{v} I$).

Put $H = (V \times V, F)$ where $\{(x,y),(x',y')\} \in F \Longleftrightarrow$ either $x = x'$ $\{y,y'\} \in E$ or $\{x,x'\} \in E$.

Given a colouring $c: V \times V \longrightarrow \{1,2\}$ either there exists $x \in V$ such that $c(\{x\} \times V) = i$ or there exists i such that for every $x \in V$ there exists y with $c((x,y)) = i$.

From this follows easily $G \xrightarrow[2]{v} H$.

Theorem 5b follows from the Erdős-Rado generalization of the classical Ramsay theorem for cardinal numbers and from the representation of finite graphs by type-graphs, see [7],[8] . This is a straightforward application of type-graphs and we ommit the proof.

This leads to the following problems (see also [2]):

Problem 2: Let G be a graph and r a cardinal number. Does there exist a graph H such that $G \xrightarrow[r]{v} H$?

Moreover, providing that G does not contain a complete graph with m vertices is it possible to choose H with the same property?

Not much is known, even the case $m = 3$ and $r = 2$ is unsolved. The purpose of this remark is to show that even dealing with vertex partitions one cannot be overoptimistic.

R e f e r e n c e s :

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