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A NOTE ON TENSOR PRODUCTS ON THE UNIT INTERVAL

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Abstract: Closedness structures on the unit interval I viewed as a thin category are considered, in view of possible applications in the calculus of fuzzy sets. The paper is concerned with the way in which continuity or discontinuity of a tensor product on I is affected by the behavior of its right adjoint.

Key words: Closedness structure, tensor product, hom-product, fuzzy set.

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Introduction. Fuzzy-set theoretists usually define the complement of a fuzzy subset $A: U \longrightarrow [0,1]$ of a universe U via the formula

$$\sim A(x) = 1 - A(x) .$$

Although the above definition ensures the validity of de Morgan formulae for fuzzy sets, one loses the useful adjunction

$$A \cap B \subset C \quad \text{iff} \quad A \subset \sim B \cup C \ ;$$

in particular, $\sim A$ is not a pseudocomplement in the lattice of all fuzzy subsets of U. This is due to the fact that the operations $x \wedge y$, $(1 - x) \vee y$ do not constitute a closedness structure on the ordered set $(I, \leq)$ viewed as a small thin category.

On the other hand, as A. Pultr showed in [4], any closedness structure on I whose unit coincides with the greatest element 1 induces a closedness structure on the category $\mathcal{F}(I)$ of all fuzzy sets which satisfies additional conditions enabling us to draw further analogies with set theory (e.g. to introduce counterparts of power-set functors). Moreover, the correspondence between structures on $(I,\leq)$ and $\mathcal{F}(I)$, respectively, is one-to-one.

Since the small category $(I,\leq)$ is skeletal, a closedness structure with unit 1 on it is completely determined by a couple $(\square,h)$ where

(i) $\square$ (the tensor product, shortly TP) is an order-preserving binary operation on I such that $(I,\square,1)$ is a commutative monoid,

(ii) h (the hom-product, shortly HP) is a binary operation on I, order-reversing in the first and order-preserving in the second variable,

(iii) the adjointness formula

(0.1) $$x \square y \leq z \quad \text{iff} \quad x \leq h(y,z)$$

holds for any $x,y,z \in I$.

By associativity of $\square$ we obtain

(0.2) $$h(x \square y,z) = h(x,h(y,z))$$

for all $x,y,z \in I$. Also observe that

(0.3) $1 = h(y,z)$ iff $1 \leq h(y,z)$ iff $y = 1 \square y \leq z$.

From (0.1) it follows that all the increasing functions $- \square x$ preserve suprema (note that preservation of sup $\emptyset$ means $x \square 0 = 0$ for any $x \in I$), the increasing functions

$h(x,-)$ preserve infima while the decreasing functions $h(-,x)$ transfer suprema to infima. A straightforward discussion of the behavior of $\Box$ and h on convergent sequences shows that, as a consequence of the monotonies, the above properties are equivalent to $\Box$ being lower-semicontinuous and h being upper-semicontinuous as real functions on $I \times I$ with the product topology.

On the other hand, since I is a complete lattice, any lower-semicontinuous operation $\Box$ on I satisfying (i) and such that $x \Box 0 = 0$ for all x can be completed to a closedness structure on I. The right adjoint h is then given by the formula

$$h(y,z) = \mathrm{Max}\{x \mid x \Box y \leq z\} .$$

We shall say that two TP's $\Box$ and $\Box'$ on I are equivalent if there exists a strictly increasing map φ of I onto itself such that

$$\varphi(x \Box y) = \varphi x \Box' \varphi y$$

holds for all $x,y \in I$. Given a TP $\Box$ on I and an automorphism φ of $(I,\leq)$ the formula

(0.4) $$x \Box^{\varphi} y = \varphi^{-1}(\varphi x \Box \varphi y)$$

defines a TP $\Box^{\varphi}$ on I equivalent to $\Box$. For the right adjoint we have

(0.5) $$h^{\varphi}(y,z) = \varphi^{-1} h(\varphi y, \varphi z)$$

As stated above, the necessary and sufficient condition for a commutative and associative operation on I with zero 0 and unit 1 to be a TP is lower-semicontinuity.

Investigating topological semigroups on manifolds with boundary, P.S. Mostert and A.L. Shields described, in particular, all topological semigroups on a compact interval with the endpoints functioning as zero and unit, respectively. Since W.M. Faucett proved in [1] that any such semigroup operation is increasing with respect to the usual order, the (I)-semigroups of Mostert and Shields coincide exactly with those TP's on I which are continuous on $I \times I$.

In § 1 we shall review some results of [1] and [3] in this direction and describe the right adjoints of some TP's including the general continuous one. It turns out that the right adjoint of a continuous TP is mostly discontinuous. Nevertheless, we may still ask what corresponds to the distinction between continuous and discontinuous TP's in terms of the hom-product. The results of § 2 indicate that such a distinction cannot be based only on the discontinuity pattern of h.

§ 1. We start with some examples of TP's. By D we denote the set of all points of $I \times I$ in which the HP h is discontinuous.

1.0 Put $x \,\square^{(0)}\, y = x \wedge y$. Then $h^{(0)}(y,z) = \begin{cases} 1 & \text{if } y \leq z, \\ z & \text{otherwise} \end{cases}$

$D^{(0)} = \{(y,y) \mid y \in [0,1[\}.$

Observe that, whatever the TP $\square$, we always have

$$x \square y \leq x \square 1 = x , \quad x \square y \leq 1 \square y = y$$

so that $\square^{(0)}$ is the greatest TP on I.

1.1 Let $\square^{(1)}$ be the usual multiplication of real numbers. Then

$$h^{(1)}(y,z) = \begin{cases} 1 & \text{if } y \leq z \\ z/y & \text{if } z < y \end{cases}, \qquad D^{(1)} = \{(0,0)\}.$$

W.M. Faucett proved in [1] that any continuous TP on I with no idempotents other than 0,1 and no nilpotents (i.e. elements $x \neq 0$ such that $x^n = 0$ for some n where the power is taken in the semigroup $(I,\square)$) is equivalent to $\square^{(1)}$.

1.2 Put $x \,\square^{(2)}\, y = \mathrm{Max}\,\{0,\ x + y - 1\}$. Then the HP

$h^{(2)}(y,z) = \mathrm{Min}\,\{1, 1 - y + z\}$ is continuous. As proved in [3], any continuous TP on I with no idempotents other than 0,1 and at least one nilpotent is equivalent to $\square^{(2)}$.

1.3. Put $x \,\square^{(3)}\, y = \begin{cases} 0 & \text{if } x + y \leq 1/2 \\ x \wedge y & \text{otherwise} \end{cases}$. Then $\square^{(3)}$ is a discontinuous TP on I with

$$h^{(3)}(y,z) = \begin{cases} 1 & \text{if } y \leq z \\ \mathrm{Max}\,\{1/2 - y, z\} & \text{otherwise} \end{cases},$$

$D^{(3)} = D^{(0)}$.

1.4. Put $x \,\square^{(4)} y = \begin{cases} 0 & \text{if } x + y \leq 1 \\ x \wedge y & \text{otherwise} \end{cases}$. Again, the product is discontinuous and we have

$$h^{(4)}(y,z) = \begin{cases} 1 & \text{if } y \leq z \\ \mathrm{Max}\ 1 - y, z & \text{otherwise} \end{cases},$$

$D^{(4)} = \{(y,y) \mid y \in]0,1[\}$.

1.5. Now we shall desdribe a construction which was shown in [3] to generate all continuous TP's from those equivalent with either $\square^{(1)}$ or $\square^{(2)}$.

Let $\{]a_\alpha , b_\alpha [\mid \alpha \in A\}$ be a countable family of disjoint open subintervals of $[0,1]$. For every $\alpha \in A$ let a TP $\square^\alpha$ on $[a_\alpha, b_\alpha]$ be given. With the family $\mathcal{F} = \{(a_\alpha , b_\alpha , \square^\alpha) \mid \alpha \in A\}$ we associate the operation $\square$ on I defined

$$(1.1) \qquad x \square y = \begin{cases} x \square^\alpha y & \text{if } (x,y) \in [a_\alpha , b_\alpha]^2 \\ x \wedge y & \text{if } (x,y) \notin \bigcup_{\alpha \in A} [a_\alpha , b_\alpha]^2 \end{cases}$$

It is easily verified that (1.1) is a correct definition of a TP on I whose set of idempotents contains $F = I \setminus \bigcup_{\alpha \in A}]a_\alpha , b_\alpha[$. Furthermore, if all $\square^\alpha$'s are continuous, so is $\square$.

On the other hand, given a continuous TP on I , denote by E the closed set of all its idempotents and consider the family $\{]a_\alpha , b_\alpha [\mid \alpha \in A\}$ of its complementary intervals. For any $\alpha \in A$ the restriction $\square^\alpha$ of $\square$ to $[a_\alpha , b_\alpha]^2$ is a continuous TP on $[a_\alpha , b_\alpha]$ with no idempotents other than a_α , b_α . Thus the ordered semigroup $([a_\alpha , b_\alpha], \leq , \square^\alpha)$ is isomorphic to either $(I, \leq , \square^{(1)})$ or $(I, \leq , \square^{(2)})$ - we shall speak of type 1 and type 2 components, respectively. Now it is easy to prove that $x \square y = x \wedge y$ whenever $(x,y) \notin \bigcup_{\alpha \in A} [a_\alpha , b_\alpha]^2$ We conclude that $\square$ coincides with the TP derived from the family $\mathcal{F} = \{(a_\alpha , b_\alpha , \square^\alpha) \mid \alpha \in A\}$ (cf. [3], Theorem B). We shall call $\mathcal{F}$ the decomposition of $\square$.

1.6. Let the TP $\square$ be obtained from a family $\mathcal{F} = \{(a_\alpha, b_\alpha, \square^\alpha) \mid \alpha \in A\}$ by construction 1.5. A straightforward computation yields the following form of the HP:

$$h(y,z) = \begin{cases} 1 & \text{if } y \leq z \\ z & \text{if } z < y \in I \setminus \bigcup_{\alpha \in A}]a_\alpha, b_\alpha[\text{ or} \\ & \quad z < a_\alpha < y < b_\alpha \text{ for some } \alpha \in A , \\ h^\alpha(y,z) & \text{if } a_\alpha \leq z < y < b_\alpha . \end{cases}$$

1.7. From (1.2) we can now derive the discontinuity pattern D of the right adjoint to a general continuous TP $\square$. Let $\mathcal{F} = \{(a_\alpha, b_\alpha, \square^\alpha) \mid \alpha \in A\}$ be the decomposition of $\square$. Assume $\square$ has at least one idempotent distinct from 0,1. Let $D_2 = \{(y, a_\alpha) \mid a_\alpha \neq 0 , \square^\alpha$ is a type 2 component, $y \in]a_\alpha, b_\alpha[\}$.

Then

(1) if there exists $\alpha \in A$ with $b_\alpha = 1$ we have

$$D = \{(y,y) \mid 0 \leq y \leq a_\alpha\} \cup D_2 ,$$

(2) otherwise

$$D = \{(y,y) \mid y \in [0,1[\} \cup D_2 .$$

§ 2.

2.1. <u>Proposition</u>. Let $\square$ be a TP on I. For any $z \in [0,1[$, the function $h(-,z)$ is continuous iff its restriction h_z to $[z,1]$ is an involutory antiisomorphism of $([z,1], \leq)$.

Proof. (1) Assume $h(-,z)$ is continuous. Since h_z is decreasing it suffices to show that $y = h_z h_z(y)$ for any $y \in [z,1]$. Next observe that

$$(2.1) \qquad y \leq h(h(y,z),z)$$

holds even without the assumption of continuity. Indeed, (2.1) is equivalent to $y \square h(y,z) \leq z$ which, by the commutativity of $\square$, amounts to $h(y,z) \leq h(y,z)$. It remains to prove the reversed inequality. Since h_z is continuous with $h_z(z) = 1$, $h_z(1) = z$, any $y \in [z,1]$ can be expressed as $y = h_z(u)$ for some $u \in [z,1]$. Then

$$y = h_z(u) \geq h_z h_z h_z(u) = h_z h_z(y)$$

where the middle inequality is obtained by applying the order-reversing function h_z to (2.1) with y replaced by u.

(2) Any antiisomorphism of $([z,1], \leq)$ is continuous. Now recall $h(y,z) = 1$ whenever $y \leq z$.

In particular, h_0 is continuous iff it is an involutory antiisomorphism of I. As for the fuzzy-set motivation, this is exactly the case when we have for any $S \subset I$, beside $h_0(\bigvee S) = \bigwedge h_0(S)$, also the other de Morgan formula $h_0(\bigwedge S) = \bigvee h_0(S)$.

For instance, the above condition is satisfied by two of the examples in § 1, namely

$$h_0^{(2)}(x) = h_0^{(4)}(x) = 1 - x .$$

Moreover, it clearly remains valid for any TP equivalent to either $\square^{(2)}$ or $\square^{(4)}$ because in that case

$$(2.2) \qquad h_0(x) = \varphi^{-1}(1 - \varphi(x))$$

where φ is an automorphism of $(I, \leq)$.

Now it is natural to ask which involutory antiisomorphisms of $(I, \leq)$ can be obtained as h_0 for some TP on I. In view of (2.2) this question is settled by the following

2.2. Proposition. For any involutory antiisomorphism f of $(I, \leq)$ there exists an automorphism φ of $(I, \leq)$ such that

$$\varphi + \varphi \circ f = 1 .$$

Proof. Given a strictly decreasing function $f: I \to I$ such that $f \circ f = \mathrm{id}$, there is exactly one point $a \in I$ with $f(a) = a$. Clearly $0 < a < 1$.

Choose any isomorphism $\psi : [0,a] \xrightarrow{\sim} [0,1/2]$ and put

$$\varphi(x) = \begin{cases} \psi(x) & \text{if } 0 \leq x \leq a \\ 1 - \psi \circ f(x) & \text{if } a \leq x \leq 1 \end{cases} .$$

Since $f(x) \leq a$ iff $x \geq a$, and $\psi(a) = 1/2 = 1 - \psi \circ f(a)$, the definition is correct and it is easy to see that φ is an automorphism of $(I, \leq)$. Finally, for any $x \in I$ we have

$$\begin{cases} x \leq a \text{ then } \varphi(x) + \varphi \circ f(x) = \psi(x) + 1 - \psi \circ f \circ f(x) = 1 \\ x \geq a \text{ then } \varphi(x) + \varphi \circ f(x) = 1 - \psi \circ f(x) + \psi \circ f(x) = 1. \end{cases}$$

Now we are going to discuss the extent to which the discontinuity pattern D of a hom-product h determines the behavior of its left adjoint $\Box$.

2.3. Proposition. If h is continuous then $\Box$ is continuous and equivalent to $\Box^{(2)}$.

Proof. (a) Since h_0 is continuous, it is an involution so that

$$x \square y = h(h(x \square y,0),0) = h(h(x,h(y,0),0)$$

holds for all $x,y \in I$, and $\square$ is continuous.

(b) Suppose $\square$ has an idempotent a with $0<a<1$. Let $x \geq a$, $y \leq a$. By continuity of $\square$ there exists $u \in I$ such that $y = a \square u$, hence

$$a \square y = a \square (a \square u) = (a \square a) \square u = a \square u = y .$$

Therefore also

$$y \leq a \square y \leq x \square y \leq 1 \square y = y .$$

Thus $h(x,b) = b$ for any $b<a$, $x \geq a$, and none of the functions h_b, $b<a$ is one-to-one which, by Proposition 2.1, contradicts the assumption on h. We conclude that $\square$ has no idempotent other than $0,1$ and is therefore equivalent to $\square^{(1)}$ or $\square^{(2)}$. The HP $h^{(1)}$ is, however, discontinuous which completes the proof.

2.4. Proposition. If h is continuous in $I^2 \setminus \{(0,0)\}$ and discontinuous at $(0,0)$ then $\square$ is continuous and equivalent to $\square^{(1)}$

Proof. (a) First we prove $\square$ continuous in all points (x,y) such that $x \square y > 0$. Take $0 < \varepsilon < x \square y$, then

$$x \square y = h_\varepsilon h_\varepsilon (x \square y) = h(h(x,h(y,\varepsilon)),\varepsilon) .$$

In the expression on the right, $x \neq 0$, $\varepsilon \neq 0$ hence $\square$ is continuous at (x,y).

(b) h_0 is discontinuous at 0 because otherwise the monotony of h and the fact that $h(0,-)$ is a constant equal to 1 would render h continuous at $(0,0)$. Thus

$$(2.3) \qquad 1 = h_0(0) > \lim_{y \to 0+} h_0(y) = a$$

We shall prove $a = 0$. Suppose that, on the contrary, $a > 0$.

First we show $h_0(x) < a$ for any $x > 0$. Let $h_0(b) = a$ and $b > 0$. Then we have $x \sqcap y = 0$ iff $y \leq a$ for any $0 < x \leq \leq b$ so that $h_0(a) \geq b$ while $h_0(x) = 0$ for any $x > a$ which contradicts the continuity of h at $(a,0)$.

Next we claim $h_0(x) > 0$ iff $x < a$. Indeed, from $h_0(b) = 0$, $b < a$ we obtain $h_0(t) \leq b$ for any $t > 0$ which contradicts (2.3). On the other hand, since $h_0(x) < a$ for $x > 0$ we have $a \sqcap x > 0$ whenever $x > 0$, hence $h_0(a) = 0$.

Finally, $a \sqcap a = a$. Indeed, the assumption $a \sqcap a < a$ yields $h_0(a \sqcap a) > 0$, and by repeated use of $h_0(a) = 0$ we obtain

$$0 < a \sqcap (a \sqcap h_0(a \sqcap a)) = (a \sqcap a) \sqcap h_0(a \sqcap a) = 0$$

which is a contradiction.

The statement $h_0(a) = 0$ together with (a) imply that the function $- \sqcap a$ is continuous in $]0,a]$. Now the argument of part (b) in the proof of Proposition 2.3 leads to discontinuity of h at (a,a).

Thus $a = 0$ and $x \sqcap y = 0$ iff $x = 0$ or $y = 0$. For any $\varepsilon > 0$ we take the open neighborhood $U = \{(s,t) \mid s \wedge t < < \varepsilon\}$ of the set $Z = \{(x,y) \mid x \sqcap y = 0\}$. We have $s \sqcap t \leq s \wedge \wedge t < \varepsilon$ for any $(s,t) \in U$ which completes the proof that $\sqcap$ is continuous.

(c) Again we can use part (b) of the proof of the preceding Proposition to show that $\sqcap$ has no other idempotents than 0,1.

Since h is discontinuous at $(0,0)$, $\square$ is equivalent to $\square^{(1)}$.

It turns out that $D = \emptyset$ and $D = \{(0,0)\}$ are the only discontinuity patterns which appear exclusively for the adjoints of continuous TP's. More exactly:

2.5. **Proposition.** For any continuous TP $\square$ on I with at least one idempotent distinct from 0 and 1 there exists a discontinuous TP $\square'$ on I with the same HP-discontinuity pattern.

Proof. (1) If the decomposition $\mathcal{F} = \{(a_\alpha, b_\alpha, \square^\alpha) \mid \alpha \in A\}$ of $\square$ contains a type 2 component $\square^\alpha$ with $b_\alpha < 1$ we can replace it by a TP $\tilde{\square}^\alpha$ on $[a_\alpha, b_\alpha]$ isomorphic to $\square^{(4)}$ and obtain a family $\mathcal{F}'$. It is easily seen from 1.4 and 1.7 that Construction 1.5 applied to the family $\mathcal{F}'$ yields a TP $\square'$ whose HP-discontinuity pattern coincides with that of $\square$. Furthermore, since $\tilde{\square}^\alpha$ is discontinuous, so is $\square'$.

(2) If there are no components of type 2 with $b_\alpha < 1$, choose an idempotent $0 < e < 1$ and a TP $\tilde{\square}$ on $[0,e]$ isomorphic to $\square^{(3)}$. Now define

$$x \,\square'\, y = \begin{cases} x \,\tilde{\square}\, y & \text{for } x,y \le e \\ x \,\square\, y & \text{for } x,y \ge e \\ x \wedge y & \text{otherwise} \end{cases}$$

Again, we obtain a discontinuous TP $\square'$ on I with the same HP-discontinuity pattern as $\square$.

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R e f e r e n c e s

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