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ON EXTENSIONS OF FULL EMBEDDINGS AND BINDING CATEGORIES

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Abstract: There are investigated extensions of full embeddings from a full subcategory of a given category to full embeddings of the whole category. Some results ensuring the existence of such an extension are stated. As an application, the following result is given: For any regular infinite cardinal m there is a three-object category M_m such that an equational class of algebras with less than m-ary operations is binding if and only if M_m can be fully embedded into it.

Key-words: Category, full embedding, Kan extension, binding category, equational class.

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Our basic situation is the following: $\mathcal{A}$ and $\mathcal{C}$ are categories, $\mathcal{M}$ is a full subcategory of $\mathcal{C}$, $K: \mathcal{M} \longrightarrow \mathcal{C}$ an inclusion functor and $T: \mathcal{M} \longrightarrow \mathcal{A}$ is a full embedding. In the first part, the construction of a functor $L_* : \mathcal{C} \longrightarrow \mathcal{A}$ extending T is recalled. The goal of this construction is the following result: If $\mathcal{M}$ is dense in $\mathcal{C}$ and cogenerates $\mathcal{C}$, L_* exists and for any $a \in \mathcal{A}$ there is a proper class of objects of $\mathcal{A}$ isomorphic with it, then L_* is a full embedding whenever a full embedding $S: \mathcal{C} \longrightarrow \mathcal{A}$ extending T exists (see [9]). Further, this part completes some techniques from [9] which make it possible to show that in the result just quoted, the existence of S can be replaced by the codensity of $\mathcal{M}$. The second part considers the basic situation for concrete categories $\mathcal{A}$ and $\mathcal{C}$. There is introduced a new functor $\overline{L} : \mathcal{C} \longrightarrow \mathcal{A}$ extending T and properties of $\overline{L}$ and L_* are dealt with. This enables us to state results concerning extensions of full embeddings in the concrete setting, which is done in the third part. The most powerful of them, Theorems 3.5 and 3.7, are in a close connection with Theorem 1 from [10] and, like this Theorem, they are originated from some considerations of [11]. The last part is devoted to applications to binding categories. A category $\mathcal{A}$ is binding if the category of graphs can be fully embedded into it (see [2]). In [11], a three-object category was found, full embeddability of which into an equational class $\mathcal{A}$ of unary algebras makes $\mathcal{A}$ to be binding and the problem was put there concerning the existence of such a small testing category for equational classes of (finitary) algebras. An affirmative solution for finitary algebras was given in [10]. In this paper we show that for any infinite regular cardinal $\mathfrak{m}$ there is a three-object category $\mathcal{M}_{\mathfrak{m}}$ testing any equational class of algebras with less than $\mathfrak{m}$-ary operations. Finally, we show that under the set axiom $\mathfrak{mon}(\mathcal{M})$ there is a monadic category which cannot be tested by a small category.

Concerning concepts of the category theory see [6].

§ 1. The functor L_*

We sketch the construction of functors $L_0, L_1, \dots, L_\alpha, \dots, L_*$ from C to A extending T and natural transformations $\lambda^{\beta,\alpha}: L_\beta \longrightarrow L_\alpha$ for $\beta < \alpha$ and $\lambda^\alpha : L_\alpha \longrightarrow L_*$ inducing the identity on K presented in [9]. $L_0 = Lan_K T$ is a pointwise left Kan extension of T along K, which means that $L_0 c$ is equal to a colimit of the functor TP, where $P: (K \downarrow c) \longrightarrow M$ is the projection of the comma category. For an isolated α, $L_\alpha c$ is a colimit of a diagram having as morphisms all morphisms of A with the domain in TM and the codomain $L_{\alpha-1} c$. Morphisms $f, g: Tm \longrightarrow L_{\alpha-1} c$ have the same domain in this diagram if and only if $L_{\alpha-1}(h)f = L_{\alpha-1}(h)g$ for any morphism $h: c \longrightarrow m$ and any $m \in M$. $\lambda_c^{\alpha-1,\alpha}$ is the component of the colimiting cone with the domain $L_{\alpha-1} c$. For a limit α, $L_\alpha c$ is a colimit of a diagram having objects $L_\beta c$ and morphisms $\lambda_c^{\beta,\beta+1}$ for $\beta < \alpha$. In both cases if $L_\alpha c$ is isomorphic to an $L_\beta c$ for $\beta < \alpha$, then we choose $L_\alpha c = L_\beta c$. If for any $c \in C$ there is γ with $L_\gamma c = L_{\gamma+1} c = L_* c$, then L_* appears. All λ are pointwise epi and commute together. Of course, any of constructed functors is given up to a natural isomorphism. Denote by $\mathcal{C}_\alpha$ or $\mathcal{C}_*$ the class of all colimits used in the construction of L_α or L_*, respectively. Put $\mathcal{D}_\alpha = \mathcal{C}_\alpha - \bigcup_{\beta<\alpha} \mathcal{C}_\beta$ for $\alpha > 0$. L_α exists whenever A has all colimits from $\mathcal{C}_\alpha$ and if A has all colimits from $\mathcal{C}_*$ and is co-well-powered, then L_* exists.

In [9], a left M-full functor $F: C \longrightarrow A$ was defined as a functor full with respect to morphisms of A domaining in $F\mathcal{M}$.

1.1. Lemma. Let $L, S : C \longrightarrow A$ be functors extending T and $\sigma : L \longrightarrow S$ a pointwise mono natural transformation inducing the identity on $\mathcal{K}$ (i.e. $\sigma K = id$). If S is left $\mathcal{M}$ -full, then L is left $\mathcal{M}$ -full, too.

For the proof it suffices to begin as in the proof of Proposition 4 from [9] and use the fact that σ is pointwise mono.

1.2. Lemma. Let $\mathcal{M}$ generate and cogenerate C and $F : C \longrightarrow A$ be a functor such that FK is faithful. Then F is faithful.

Proof: For any $g \neq g' : c \longrightarrow c'$ of C one can find m, $m' \in \mathcal{M}$, $f : m \longrightarrow c$ and $h : c' \longrightarrow m'$ with $hgf \neq hg'f$. Since FK is faithful, F must be faithful.

1.3. Definition. Let $F : C \longrightarrow A$ be a functor, $\mathcal{X}$ a subcategory of C . We say that $\mathcal{X}$ left F -generates C if for a morphism $f : Fc \longrightarrow Fc'$ of A , $f = F(f')$ holds for a morphism $f' : c \longrightarrow c'$ of C whenever for any $x \in \mathcal{X}$ and any morphism $h : x \longrightarrow c$ of C there exists a morphism $h' : x \longrightarrow c'$ of C such that $F(h') = fF(h)$.

This definition generalizes the concept of an inductively generating subcategory $\mathcal{X}$ of a concrete category (C, F) (it means that $F : C \longrightarrow \mathcal{E}ns$ is a faithful functor). Dually, right F -generation extends projective generation.

1.4. Lemma. If $\mathcal{X}$ left F -generates C and F' is naturally isomorphic to F , then $\mathcal{X}$ left F'-generates C .

Proof is evident.

1.5. Lemma. Let $F: \mathcal{C} \longrightarrow \mathcal{A}$, $G: \mathcal{A} \longrightarrow \mathcal{B}$ be functors and $\mathcal{X}$ a subcategory of $\mathcal{C}$. If $\mathcal{X}$ left GF-generates $\mathcal{C}$ and G is faithful, then $\mathcal{X}$ left F-generates $\mathcal{C}$.

Proof is evident.

1.6. Lemma. Let $L: \mathcal{C} \longrightarrow \mathcal{A}$ be a left $\mathcal{M}$-full functor. Then L is full iff $\mathcal{M}$ left L-generates $\mathcal{C}$.

Proof is evident.

1.7. Proposition: Let $L: \mathcal{C} \longrightarrow \mathcal{A}$ be a functor extending T and $\mathcal{M}$ left L-generate $\mathcal{C}$. Then L is right $\mathcal{M}$-full.

Proof: If $c \in \mathcal{C}$, $m' \in \mathcal{M}$ and $f: c \longrightarrow m'$ is a morphism of $\mathcal{C}$, then the fulness of T implies that $fL(h) = L(h')$ for any $m \in \mathcal{M}$ and $h: m \longrightarrow c$. Therefore $f = L(f')$.

A family $\{f_i\}$ of morphisms of $\mathcal{A}$ with the same codomain a is called jointly epi whenever $f = g$ for any parallel pair of morphisms of $\mathcal{A}$ with the domain a such that $ff_i = gf_i$ for any i.

1.8. Proposition: Let $\mathcal{X}$ be dense in $\mathcal{C}$, $I: \mathcal{X} \longrightarrow \mathcal{C}$ the inclusion, $F: \mathcal{C} \longrightarrow \mathcal{A}$ a faithful functor and $\{F(h)/h: m \longrightarrow c, m \in \mathcal{M}\}$ a jointly epi family in the full subcategory of $\mathcal{A}$ determined by $F\mathcal{C}$ for any $c \in \mathcal{C}$. Then $\mathcal{X}$ left F-generates $\mathcal{C}$.

Proof: Following the first part of the proof of Proposition 3 from [9] it suffices to take for f from 1.3 the natural transformation $\tau: \mathcal{C}(I-, c) \longrightarrow \mathcal{C}(I-, c')$ defined by $F(\tau_x(h)) = fF(h)$, to use the density of $\mathcal{X}$ for finding $f': c \longrightarrow c'$ in $\mathcal{C}$ with $F(f'h) = fF(h)$ for any

$h: m \to c$, $m \in M$ and to realize that $F(f') = f$ for $\{F(h)\}$ is jointly epi.

1.9. Corollary. Let M be dense in C and $L: C \to A$ a faithful functor extending T such that the unique natural transformation $\sigma: L_0 \to L$ inducing the identity on K is pointwise epi. Then M left L-generates C.

Proof: In the same way as in the last six lines of the proof of Proposition 3 from [9] it can be proved that $\{L(h) / h: m \to c, m \in M\}$ is a jointly epi family in A for any $c \in C$. Hence M left L-generates C by 1.8.

Since 1.2 holds and $\lambda^{0,\alpha}$, λ^{α} are pointwise epi, functors $L_0, \dots, L_\alpha, \dots, L_*$ fulfil suppositions of 1.9 whenever M cogenerates C.

1.10. Corollary. Let M be dense in C, $F: C \to A$ a faithful and right M-full functor and FM cogenerate the full subcategory of A determined by FC. Then M left L-generates C.

Proof: Let $f \neq g: Fc \to Fc'$ such that $fF(h) = gF(h)$ for any $h: m \to c$, $m \in M$. There is $m' \in M$ and $k: Fc' \to Fm'$ with $kf \neq kg$. Since F is right M-full, $kf = F(k_1)$ and $kg = F(k_2)$. There is $m \in M$ and $h: m \to c$ with $k_1 h \neq k_2 h$ because M generates C. Hence $kfF(h) = F(k_1 h) \neq F(k_2 h) = kgF(h)$, which is a contradiction.

1.11. Theorem. Let M be dense and codense in C, L_* exist and let any $a \in A$ have a proper class of objects of A isomorphic with it. Then L_* is a full embedding.

Proof: L_* is faithful by 1.2 and right M-full by 1.9 and 1.7. Further, $\{L_*(h) / h: c \to m, m \in M\}$ is jointly mono by the construction of $L_* c$ for any $c \in C$. The dual of 1.8 ensures that $\mathcal{M}$ right L_*-generates C. Now, L_* is full by the dual of 1.6. Since A has enough isomorphic copies of each of its object, L_* can be chosen to be an embedding (the big axiom of choice is used here).

§ 2. Concrete setting

We add to our basic situation that (C, U) and (A, V) are concrete categories. We shall define a functor $\bar{L}$ which turns out to be useful.

Let $c \in C$ and $\sim$ be the equivalence on $VL_0 c$ defined as follows: $x \sim y$ for $x, y \in VL_0 c$ iff $VL_0(h)(x) = VL_0(h)(y)$ for any $h : c \longrightarrow m$ in C and any $m \in \mathcal{M}$. Let $\bar{\lambda}_c : L_0 c \longrightarrow \bar{L} c$ be a morphism of A with the following property: $V(\bar{\lambda}_c)(x) = V(\bar{\lambda}_0)(y)$ for any $x, y \in VL_0 c$ with $x \sim y$ and if $f : L_0 c \longrightarrow a$ is a morphism of A with $V(f)(x) = V(f)(y)$ for any such x, y, then there is a unique $k : \bar{L} c \longrightarrow a$ in A such that $k \bar{\lambda}_c = f$. Let $g : c \longrightarrow d$ be a morphism of C and $x, y \in VL_0 c$ such that $x \sim y$. Then $VL_0(g)(x) \sim VL_0(g)(y)$ and let $\bar{L}(g)$ be a unique morphism of A such that $\bar{L}(g) \bar{\lambda}_c = \bar{\lambda}_d L_0(g)$. Then $\bar{L} : C \longrightarrow A$ is a functor and $\bar{\lambda} : L_0 \longrightarrow \bar{L}$ a natural transformation. Moreover, $\bar{L}$ is an extension of T and $\bar{\lambda}$ induces the identity on $\mathcal{K}$ because $x, y \in VL_0 m$, $m \in \mathcal{M}$ and $x \sim y$ implies that $x = VL_0(id_m)(x) = VL_0(id_m)(y) = y$. Again, $\bar{L}$ is defined up to a natural isomorphism.

2.1. Definition. Let $D : S \longrightarrow A$, $F : A \longrightarrow X$ be functors and $a \in A$ a colimit of D with the colimiting cone

$\tau : D \longrightarrow a$. We say that F compresses the colimit a if $\{F(\tau_s) / s \in S\}$ is jointly epi.

Of course, if F preserves colimits, then it compresses them.

2.2. <u>Lemma</u>. Let D, F, a, τ be as in the definition and, in addition, let FD have a colimit $x \in \mathcal{X}$. Then F compresses a iff the induced morphism $k : x \longrightarrow Fa$ is epi.

Proof is evident.

We say that an object e of C is nearly 1-free whenever there exists a natural monotransformation $\mu : U \longrightarrow C(e,-)$. We recall that for a 1-free object μ is requested to be iso. Examples of nearly 1-free objects which are not 1-free are supplied by concrete categories with constants because any object e of such a category is nearly 1-free with μ_c assigning to each $x \in Uc$ the constant morphism with the value x .

2.3. <u>Lemma</u>. Any nearly 1-free object is a generator of C .

<u>Proof</u>: Let $f, g : c \longrightarrow d$ be morphisms of C and $fh = gh$ for any $h : e \longrightarrow c$. Hence $\mu_d U(f) = C(e,f)\mu_c = C(e,g)\mu_c = \mu_d U(g)$ and $f = g$ because U is faithful and μ_d mono.

2.4. <u>Proposition</u>: Let $\mathcal{X}$ be a subcategory of C containing a nearly 1-free object e of C . If $\mathcal{X}$ inductively generates C , then $\mathcal{X}$ is dense in C . If $\nu_{Uc}(x) = U(\mu_c(x))$, $x \in Uc$, $c \in C$ are components of a natural transformation $\nu : J \longrightarrow \mathcal{E}ns(\mathcal{U}e, J-)$ where $J : \mathcal{U}C \longrightarrow \mathcal{E}ns$ is the inclusion, then the converse implication holds.

Proof: Let X inductively generate C, $I : X \longrightarrow C$ be the inclusion, $c, d \in C$ and $\tau : C(I-,c) \longrightarrow C(I-,d)$ a natural transformation. Let $x_i \in X$, $g_i : x_i \longrightarrow c$ in C and $z_i \in \mathcal{U}x_i$ for $i = 1, 2$ such that

(1) $$\mathcal{U}(g_1)(z_1) = \mathcal{U}(g_2)(z_2) .$$

Since μ is natural, for $i = 1, 2$ it holds

(2) $$C(e, g_1)\mu_{x_i} = \mu_c \mathcal{U}(g_i) ,$$

(3) $$C(e, \tau_{x_i}(g_i))\mu_{x_i} = \mu_d \mathcal{U}(\tau_{x_i}(g_i)) .$$

Similarly, the naturality of τ implies

(4) $$\tau_e C(\mu_{x_i}(z_i), c) = C(\mu_{x_i}(z_i), d)\tau_{x_i} .$$

Evaluating (2) at z_i and using (1), we get $g_1\mu_{x_1}(z_1) = g_2\mu_{x_2}(z_2)$. Further, by evaluation of (4) at g_i we obtain $\tau_{x_1}(g_1)\mu_{x_1}(z_1) = \tau_{x_2}(g_2)\mu_{x_2}(z_2)$. Finally, the evaluation of (3) at z_i yields $\mu_d(\mathcal{U}(\tau_{x_1}(g_1))(z_1)) = \mu_d(\mathcal{U}(\tau_{x_2}(g_2))(z_2))$, and thus $\mathcal{U}(\tau_{x_1}(g_1))(z_1) = \mathcal{U}(\tau_{x_2}(g_2))(z_2)$ because μ is mono.

This observation makes it possible to find a mapping $f : \mathcal{U}c \longrightarrow \mathcal{U}d$ such that $f\mathcal{U}(g) = \mathcal{U}(\tau_x(g))$ for any $x \in X$ and $g : x \longrightarrow c$. Since X inductively generates C, there is a morphism $f' : c \longrightarrow d$ of C with $\mathcal{U}(f') = f$. Therefore $\tau = C(I-, f')$ and the density of X is verified.

The rest of the proof follows from 1.8 using the computation of the proof of 2.3 applying to ν instead of μ.

ν is natural if e is 1-free or if μ is defined by constants. Thus 2.4 generalizes Lemma 3 from [9].

2.5. <u>Proposition</u>: Let A have colimits from $\mathcal{C}_1$ and TM contain a nearly 1-free object e of $L_0 c$. Then $\overline{L} = L_1$. In addition, if V compresses colimits from D_1, then $L_* = L_1$.

<u>Proof</u>: Let $c \in C$, $x, y \in VL_0 c$ and $x \sim y$. Since $VL_0(h)(x) = VL_0(h)(y)$ for any $h: c \longrightarrow m$, $m \in M$ and μ is natural, $L_0(h)\mu_{L_0 c}(x) = \mu_{Tm}(VL_0(h)(x) = \mu_{Tm}(VL_0(h)(y)) = L_0(h)\mu_{L_0 c}(y)$ for any h. Therefore $\lambda_c^{0,1}\mu_{L_0 c}(x) = \lambda_c^{0,1}\mu_{L_0 c}(y)$ and further $V(\lambda_c^{0,1})(x) = V(\lambda_c^{0,1})(y)$ because μ is natural and mono (and therefore pointwise mono because Ens is cocomplete).

Let $f: L_0 c \longrightarrow a$ be a morphism of A with $V(f)(x) = V(f)(y)$ for any $x, y \in VL_0 c$, $x \sim y$. Let $m \in M$ and $f', g': Tm \longrightarrow L_0 c$ be morphisms of A such that $L_0(h)f' = L_0(h)g'$ for any $h: c \longrightarrow m$, $m \in M$. Hence $V(f')(z) \sim V(g')(z)$ for any $z \in TM$ and thus $ff' = fg'$. By the definition of $\lambda_c^{0,1}$ there is a unique $k: L_1 c \longrightarrow a$ such that $k\lambda_c^{0,1} = f$. Thus, $\lambda_c^{0,1} = \overline{\lambda}$ and $\overline{L} = L_1$.

If V compresses colimits from D_1, then $V\lambda_c^{0,1}$ is epi. Let $m \in M$ and $f, g: Tm \longrightarrow L_1 c$ such that $L_1(h)f) = L_1(h)g$ for any $h: c \longrightarrow m$, $m \in M$. For any $z \in VTm$ there are $x, y \in VL_0 c$ such that $V(\lambda_c^{0,1})(x) = V(f)(z)$ and $V(\lambda_c^{0,1})(y) = V(g)(z)$. The naturality of $\lambda_c^{0,1}$ implies that $x \sim y$ and therefore $V(\lambda_c^{0,1})(x) = V(\lambda_c^{0,1})(y)$. Thus $f = g$

and $L_* = L_1$.

A close connection between L_* and $\overline{L}$ is in a general case, too, because $\overline{L}: C \longrightarrow A$ can be obtained as a restriction of $L_*: C_e \longrightarrow A_e$, where C_e and A_e arise from C and A by a suitable addition of a 1-free object e .

2.6. <u>Proposition</u>: Let $T\mathcal{M}$ contain a nearly 1-free object e of A, $\alpha > 0$ be an ordinal, A have colimits from $\mathcal{C}_\alpha$ and V compress colimits from $\mathcal{D}_\alpha$. Then $L_* = L_\alpha$.

Proof follows from similar computations as in the proof of 2.5.

2.7. <u>Lemma</u>. Let $T\mathcal{M}$ contain a nearly 1-free object e of $L_0 C$ and V compress colimits from $\mathcal{C}_0$. Then any natural transformation $\sigma: L_0 \longrightarrow S$ inducing the identity on K into a left $\{e\}$-faithful functor S is pointwise mono.

<u>Proof</u>: Let $c \in C$ and $x_1, x_2 \in VL_0 c$ with $V(\sigma_c)(x_1) = V(\sigma_c)(x_2)$. Since V compresses colimits from $\mathcal{C}_0$, there are $m_i \in \mathcal{M}$, $f_i : m_i \longrightarrow c$ and $z_i \in VTm_i$ with $VL_0(f_i)(z_i) = x_i$ for $i = 1, 2$. Since T is full, there exist morphisms $\varkappa_i$ of $\mathcal{M}$ domaining in e with $T(\varkappa_i) = \mu_{Tm_i}(z_i)$ for $i = 1, 2$. It holds $S(f_1\varkappa_1) = \sigma_c L_0(f_1\varkappa_1) = \sigma_c L_0(f_1)\mu_{Tm_1}(z_1) = \sigma_c \mu_{L_0 c}(VL_0(f_1)(z_1)) = \sigma_c \mu_{L_0 c}(x_1) = \mu_{Sc}(V(\sigma_c)(x_1)) = \mu_{Sc}(V(\sigma_c)(x_2)) = S(f_2\varkappa_2)$ by the naturality of σ and α . The left $\{e\}$-faithfulness of S yields $f_1\varkappa_1 = f_2\varkappa_2$. Further, $\mu_{L_0 c}(x_1) = \mu_{L_0 c}(VL_0(f_1)(z_1)) = L_0(f_1)\mu_{Tm_1}(z_1) = L_0(f_1)T(\varkappa_1) = L_0(f_2\varkappa_2) = \mu_{L_0 c}(x_2)$. Since μ is mono, $x_1 = x_2$ and therefore σ is pointwise mono.

2.8. <u>Theorem</u>. Let $T\mathcal{M}$ contain a nearly 1-free object of $L_0 C$, let A have and V compress colimits from $\mathscr{C}_0$. Let $\mathcal{M}$ be dense in C, any $a \in A$ have a proper class of objects of A isomorphic with it and let there exist a full embedding S extending T. Then L_0 is a full embedding.

Proof: L_0 is faithful by Proposition 1 from [9], and left $\mathcal{M}$-full by 2.7 and 1.1. Therefore L_0 is full by 1.9 and 1.6.

This result generalizes Proposition 4 from [9].

2.9. Definition. We call a functor $L: C \longrightarrow A$ V-covered (with respect to $\mathcal{M}$) if the family $\{VL(f) / m \in \mathcal{M}, f: m \longrightarrow c\}$ is jointly epi.

2.10. Lemma. Let A have and V compress colimits from $\mathscr{C}_\alpha$ or $\mathscr{C}_*$. Then L_α or L_* resp. exists and is V-covered.

Proof: If V compresses colimits from $\mathscr{C}_0$, then $V(\lambda_c^{Ord})$ is epi for any ordinal α and any $c \in C$. Thus there is only a set of λ_c^{Ord} and therefore L_* exists. Further, $\{VL_0(f) / f: m \longrightarrow c, m \in \mathcal{M}\}$ is jointly epi because V compresses the colimit $L_0 c = Colim((K \downarrow c) \xrightarrow{P} \mathcal{M} \xrightarrow{T} A)$ having $\{L_0(f) / f: m \longrightarrow c, m \in \mathcal{M}\}$ as components of the colimiting cone for any $c \in C$. Since V compresses colimits from $\mathscr{C}_\alpha$ ($\mathscr{C}_*$) and any λ is a natural transformation, the assertion holds.

If $f_i : a_i \longrightarrow a$ is a family of monics of A having a limit in A, then a component of the limiting cone with the codomain a is monic which is called an intersection of f_i.

2.11. Lemma. Let A have and V preserve finite limits and arbitrary intersections. If A has coequalizers and colimits from $\mathcal{C}_0$, then $\bar{L}$ exists and if V compresses them, then $\bar{L}$ is V-covered.

Proof: Let $c \in C$ and $i : \sim \longrightarrow VL_0c \times VL_0c$ be the monic defined by the equivalence $\sim$. Let $d : b \longrightarrow L_0c \times L_0c$ be the intersection of all monics $j : x \longrightarrow L_0c \times L_0c$ of A such that i can be factorized through Vj. Since V preserves intersections, $i = V(d)k_1$ for a unique mapping k_1. Let $p_1, p_2 : L_0c \times L_0c$ be projections and p a coequalizer of p_1d and p_2d. It holds $V(p)V(p_1)i = = V(p)V(p_2)i$. Let $f : L_0c \longrightarrow a$ be a morphism of A with $V(f)V(p_1)i = V(f)V(p_2)i$ and g an equalizer of fp_1d and fp_2d. Since V preserves finite limits, $V(g)$ is an equalizer of $V(fp_1d)$ and $V(fp_2d)$. Since $V(fp_1d)k_1 = = V(fp_1)i = V(fp_2)i = V(fp_2d)k_1$, there is a unique k_2 with $V(g)k_2 = k_1$. Hence $V(dg)k_2 = i$ and the definition of d yields that $V(g)$ is iso. Thus $V(fp_1d) = V(fp_2d)$ and $fp_1d = fp_2d$. There is a unique morphism k of A such that $kp = f$. Hence $\bar{\lambda}_c = p$.

Thus $\bar{L}$ exists, when A has coequalizers and colimits from $\mathcal{C}_0$. If V compresses them, then $V(\bar{\lambda}_c)$ is epi for any $c \in C$ and L_0 is V-covered by 2.10 and therefore $\bar{L}$ is V-covered.

Conversely, if $\bar{L}$ exists, A has and V preserved kernel pairs, then $\bar{\lambda}_c$ is a coequalizer for any $c \in C$.

2.12. Lemma. Let $\bar{L}$ be V-covered. Then $\{VL(h) / h : : c \longrightarrow m, m \in \mathcal{M}\}$ is jointly monic for any $c \in C$.

Proof: Using the naturality of $\bar{\lambda}$ it can be proved that $V\bar{\lambda}$ is epi for $\bar{L}$ is V-covered and then, by the definition of $\bar{L}$, that Lemma holds.

§ 3. Fulness in concrete setting

3.1. Proposition: Let $L: C \longrightarrow A$ be a faithful functor extending T such that $VL \cong U$ and M inductively generate C. Then L is right M-full.

Proof follows from 1.4, 1.5 and 1.7.

(A,V) has the property of transfer if for every object a of A and for every bijection f from Va onto an arbitrary set x there is an isomorphism f' of A such that $V(f') = f$. (C,U) has the property of unicity, if every isomorphism f of C such that $V(f) = id$ is an identity (see [8]). A full embedding $L: C \longrightarrow A$ such that $VL = U$ is called a realization (see [7]).

3.2. Theorem. Let (C,U) have the property of unicity, (A,V) of transfer, M inductively and projectively generate C. Let $L: C \longrightarrow A$ be a faithful functor extending T and let there exist a natural isomorphism between U and VL inducting an identity on K. Then there is a realization $C \longrightarrow A$ extending T naturally isomorphic to L.

Proof: L is full by 1.4, 1.5, the dual of 3.1 and by 1.6. The rest of the proof follows from [8], Lemma 1.5.

3.3. Theorem. Let (C,U) have the property of unicity, (A,V) of transfer, A have and V compress colimits from $\mathcal{C}_0$. Let M inductively and projectively generate C, cogenerate C and contain a 1-free object e of C such that Te is a 1-free object of A. Let $\mu : U \longrightarrow C(e,-)$, $\bar{\mu} : V \longrightarrow A(Te,-)$ be corresponding natural isomorphisms, $\beta : C(e,-) \longrightarrow A(Te, L_0 -)$ a natural transformation defined by $\beta_c(f) = L_0(f)$ and $(\bar{\mu}^{-1} . L_0\beta . \mu)X$ be an identity.

Then L_0 is a realization.

Proof: Since M cogenerates C, L_0 is faithful by 2.3 and 1.2 and therefore β is mono. Hence the composition $\alpha = \bar{\mu}^{-1} . L_0\beta . \mu : U \longrightarrow VL_0$ is a natural monotransformation. βX is iso because T is full and $\{VL_0(f)/m \in M, f : m \longrightarrow c\}$ is jointly epi by 2.10. Therefore α is epi. Hence α is a natural isomorphism and L_0 can be chosen as a realization by 3.2.

3.4. Theorem. Let (C, U) have the property of unicity, (A, V) of transfer, M inductively and projectively generate C, $\{U(h)/h : c \longrightarrow m, m \in M\}$ be jointly monic and $\{U(f)/f : m \longrightarrow c, m \in M\}$ jointly epi for any $c \in C$. Let T be a realization and $L : C \longrightarrow A$ a V-covered functor extending T such that $\{VL(h)/h : c \longrightarrow m, m \in M\}$ is jointly monic for any $c \in C$.

Then there is a realization $C \longrightarrow A$ extending T naturally isomorphic to L.

Proof: Since $\{U(h)/h : c \longrightarrow m, m \in M\}$ or $\{U(f)/f : m \longrightarrow c, m \in M\}$ is jointly monic or epi, M cogenerates or generates C, respectively. Hence L is faithful by 1.2. Let $c \in C$ and define a mapping $\alpha_c : Uc \longrightarrow VLc$ as follows: $\alpha_c(x) = VL(f)(z)$ if $x = U(f)(z)$ for $f : m \longrightarrow c$, $m \in M$,

$x \in \mathcal{U}m$. Indeed, α_c is a mapping because any x is equal to a $\mathcal{U}(f)(x)$ and $\mathcal{U}(f_1)(x_1) = \mathcal{U}(f_2)(x_2)$ for $f_i : m_i \longrightarrow c$, $m_i \in \mathcal{M}$, $x_i \in m_i$ implies that $VL(hf_1)(x_1) = VT(hf_1)(x_1) = \mathcal{U}(hf_1)(x_1) = \mathcal{U}(hf_2)(x_2) = VL(hf_2)(x_2)$ for any $h : c \longrightarrow m, m \in \mathcal{M}$ for T is a realization and therefore $VL(f_1)(x_1) = VL(f_2)(x_2)$ because $\{VL(h) / h : c \longrightarrow m,\ m \in \mathcal{M}\}$ is jointly monic. Let $x_1 \neq x_2 \in \mathcal{U}c$, $x_i = \mathcal{U}(f_i)(x_i)$. Since $\{\mathcal{U}(h) / h : c \longrightarrow m,\ m \in \mathcal{M}\}$ is jointly monic, there is $h : c \longrightarrow m$ with $VL(hf_1)(x_1) = \mathcal{U}(hf_1)(x_1) \neq \mathcal{U}(hf_2)(x_2) = VL(hf_2)(x_2)$. Hence α_c is mono. Since L is V-covered, α_c is epi. It is easy to compute that α_c are components of a natural isomorphism $\alpha : \mathcal{U} \longrightarrow VL$. Thus the Theorem follows from 3.2.

A full embedding $L : C \longrightarrow A$ is called a pseudorealization in [8] when $\mathcal{U}c \subseteq VLc$ for any $c \in C$ and these inclusions are components of a natural transformation $\mathcal{U} \longrightarrow VL$.

3.5. Theorem. Let the suppositions from the first sequence of 3.4 hold. Let T be a pseudorealization, $\overline{L}$ exist and be V-covered. Let for any $m \in \mathcal{M}$, $c \in C - \mathcal{M}$ and $g : m \longrightarrow c$ in C there exist $n \in \mathcal{M}$ and $h_0 : c \longrightarrow n$ with the following properties:

a) for any $y \in \mathcal{U}n - \mathcal{U}(h_0 g)(\mathcal{U}m)$ there are $s, s' : n \longrightarrow n$ such that $ss'h_0 g = s'h_0 g$, $\mathcal{U}(s')(y) = y$ and $\mathcal{U}(s)(y) \neq y$;

b) for any $m' \in \mathcal{M}$ and $h : c \longrightarrow m'$ $\{\mathcal{U}(t) / t : m' \longrightarrow n, thg = t'h_0 g$ for a $t' : n \longrightarrow n\}$ is jointly monic.

Then $\overline{L}$ is a pseudorealization.

Proof: Put $Wa = \{x \in Va / x = V(f)(x), m \in \mathcal{M}, f : Tm \longrightarrow a$ and $x \in \mathcal{U}m\}$ and $W(g) = V(g)/Wa$ for any $g : a \longrightarrow b$ in

A . Evidently $W: A \longrightarrow \mathcal{E}ns$ is a functor and $WT = \mathcal{U}$ because T is a pseudorealization. Let A' be a full subcategory of A consisting of all objects of A isomorphic with an object from $\bar{L}C$ and W' be the restriction of W onto A'. For proving that W' is faithful it suffices to show that W' is faithful on $\bar{L}C$. Let $c, d \in C$ and $g_1 \neq g_2 : \bar{L}c \longrightarrow \bar{L}d$. Since $\bar{L}$ is V-covered, there exists $m \in \mathcal{M}$ and $f: m \longrightarrow c$ such that $g_1 \bar{L}(f) \neq g_2 \bar{L}(f)$. By 2.12 there is $m' \in \mathcal{M}$ and $h: d \longrightarrow m'$ such that $\bar{L}(h) g_1 \bar{L}(f) \neq \bar{L}(h) g_2 \bar{L}(f)$. Further, $\bar{L}(h) g_i \bar{L}(f) = \bar{L}(r_i)$ for $r_i : m \longrightarrow m'$, $i = 1, 2$ and $\mathcal{U}(r_1) \neq \mathcal{U}(r_2)$. Thus there is $z \in \mathcal{U}m$ such that $V\bar{L}(r_1)(z) \neq V\bar{L}(r_2)(z)$, which implies that $W'(g_1) \neq W'(g_2)$.

If we show that $\bar{L}$ is W-covered, then the Theorem will follow from 3.4 applied to the concrete category (A', W') instead of to (A, V) . Consider $c \in C$ and $x \in W'\bar{L}c$. There is $m \in \mathcal{M}$, $g: m \longrightarrow c$ and $z \in VTm$ with $x = V\bar{L}(g)(z)$ because $\bar{L}$ is V-covered. Take n and h_0 for this g . Since $x \in W'\bar{L}c$, $y = V\bar{L}(h_0)(x) \in \mathcal{U}n$. Further $y \in V\bar{L}(h_0 g)(\mathcal{U}m)$ because otherwise taking s and s' for this y we get a contradiction. Namely, $y = V\bar{L}(s')(y) = V\bar{L}(s' h_0 g)(z) = V\bar{L}(s s' h_0 g)(z) = V\bar{L}(s s')(y) \neq y$. Hence $y = V\bar{L}(h_0 g)(w)$ for some $w \in \mathcal{U}m$. Suppose that $V\bar{L}(h)(x) \neq V\bar{L}(hg)(w)$ for an $h: c \longrightarrow m'$ and $m' \in \mathcal{M}$. By b) there are $t: m' \longrightarrow n$, $t': n \longrightarrow n$ with $V\bar{L}(t' h_0 g)(w) = V\bar{L}(thg)(w) \neq V\bar{L}(th)(x) = V\bar{L}(thg)(z) = V\bar{L}(t' h_0 g)(z) = V\bar{L}(t' h_0)(x)$, which is a contradiction. Hence $x = V\bar{L}(g)(w)$ by 2.12 and the proof is accomplished.

3.6. <u>Remarks</u>: 1) Conditions ensuring in 3.4 and 3.5 that $\bar{L}$ exists and is V-covered, are yielded by 2.11 or 2.5 and 2.10.

2) If $\{U(f)/f: m \longrightarrow c, m \in M\}$ is not always jointly epi, then $U'c = \bigcup\{U(f)(Um)/f: m \longrightarrow c,\ m \in M\}$ defines a new concrete category (C, U') which shares with (C, U) all other suppositions of 3.4 or 3.5.

3) Condition b) can be replaced by the following one: $V\bar{L}(h_0)(x_1) = V\bar{L}(h_0)(x_2)$ for $x_1, x_2 \in W\bar{L}c$ implies that $V\bar{L}(h)(x_1) = V\bar{L}(h)(x_2)$ for any $h: c \longrightarrow m$ and $m \in M$.

3.7. <u>Theorem</u>. Let (C, U) have the property of unicity, (A, V) of transfer, M inductively and projectively generate C, cogenerate C, let A have and V compress colimits from $\mathscr{C}_*$. Let M contain a 1-free object e of C and for any $m \in M, c \in C - M$ and $g: m \longrightarrow c$ of C there exist a cogenerator $n \in M$ of C and $h_0: c \longrightarrow n$ such that

a_1) for any permutation t' of Un interchanging precisely two elements of Un there is a morphism $t: n \longrightarrow n$ of C such that $U(t) = t'$,

a_2) $\mathrm{card}\,(Un - U(h_0 g)(Um)) > 1$

and b) from 3.5 hold.

Then L_* is a pseudorealization.

This Theorem is a slight generalization of Theorem 1 from [10] (use 2.10 and 2.4). Conditions a_1) and a_2) imply a) (see [10]).

§ 4. Binding categories

If $\mathfrak{m}$ is a regular cardinal, then a category J is called $\mathfrak{m}$-filtered when J is not empty, to any family $\{j_i\}$ of less than $\mathfrak{m}$ objects of M there is $j \in J$ and morphisms $j_i \longrightarrow j$ for any i and for any family $\{f_i\}$ of less than $\mathfrak{m}$ parallel morphisms of J there is a morphism f of J equalizing all f_i. It follows that any diagram in J having less than $\mathfrak{m}$ morphisms is a base of a cone in J.

4.1. Lemma. Let $\mathcal{M}' = \{m \in \mathcal{M} /$ there is $c \in C - \mathcal{M}$ and a morphism $m \longrightarrow c$ in $C\}$ be non-empty and contain any colimit in C of a diagram in $\mathcal{M}'$ having less than $\mathfrak{m}$ morphisms. Then the comma category $(K \downarrow c)$ is $\mathfrak{m}$-filtered for any $c \in C$.

Proof is evident.

Let $\mathfrak{m}$ be a regular infinite cardinal, $\underline{\mathfrak{m}}$ the complete graph (with loops) having $\mathfrak{m}$ vertices and $\underline{1}$, $\underline{2}$ or $\underline{4}$ the complete graph without loops having one, two or four vertices respectively. Let $C_{\mathfrak{m}}$ be the full subcategory of the category of undirected graphs composed of all connected 3-colourable graphs and the graph $m_{\mathfrak{m}}$ having $\underline{\mathfrak{m}}$ and $\underline{4}$ as components and $M_{\mathfrak{m}}$ the full subcategory of $C_{\mathfrak{m}}$ determined by $\underline{1}$, $m_{\mathfrak{m}}$ and the graph $p_{\mathfrak{m}}$ having $\mathfrak{m}$ copies of $\underline{2}$ as components. $C_{\mathfrak{m}}$ is binding for the category of connected 3-colourable graphs (see[11]). Let U be the usual forgetful functor of the category of graphs. Any restriction of U onto a full subcategory will be denoted by the same letter.

4.2. Theorem. Let (A, V) have the property of transfer, A have and V preserve finite limits and arbitrary intersections. Let $\varkappa$ be a regular infinite cardinal, let A have and V compress coequalizers and small $\varkappa$-filtered colimits.

Then A is binding if and only if $M_\varkappa$ can be fully embedded into it.

Proof: If A is binding, then any small category can be fully embedded into it (see [3]). Let $M_\varkappa$ be fully embedded into A. $\{U(h)/h : c \longrightarrow \underline{4}\}$ is jointly monic for any $c \in C_\varkappa$ (see [11], (a) of Lemma 1). Thus, there is a pseudorealization $T : M_\varkappa \longrightarrow A$ by Theorem 1.8 from [8] because $\underline{1}$, $n_\varkappa \in M_\varkappa$ and $\underline{4}$ satisfies a_1) from 3.6. $M_\varkappa$ projectively generates $C_\varkappa$, for $n_\varkappa$ does (see [11], (b) of Lemma 1) and inductively generates $C_\varkappa$, for $p_\varkappa$ does. Since there is no morphism from $\underline{4}$ to a 3-colourable graph, $(K \downarrow c)$ is $\varkappa$-filtered essentially by 4.1, which can be verified by consideration of coproducts of less than $\varkappa$ objects and coequalizers, because M need not contain coproducts of $\underline{1}$, for those are subgraphs of $p_\varkappa$ and any morphism from such a subgraph into a connected graph with at least two vertices can be extended to the whole $p_\varkappa$. Following 2.11, $\overline{L}$ exists and is V-covered. Finally, let $m \in M_\varkappa$, $c \in C_\varkappa - M_\varkappa$ and $g : m \longrightarrow c$ be a morphism of $C_\varkappa$. Take $n = n_\varkappa$ and $h_0 :$ $: c \longrightarrow n_\varkappa$ such that $U(h_0)(Uc) \subseteq \varkappa$ and $U(h_0)/U(g)(Um)$ is injective. Then a) holds because a_1) and a_2) from 3.7 are satisfied. b) holds for $\{U(t)/t : m' \longrightarrow \varkappa\}$ is jointly monic for any $m' \in M$ and hg can be factorized through $h_0 g$ for any $h : c \longrightarrow \kappa$. Therefore Theorem 4.2 follows from 3.5.

4.3. <u>Corollary</u>. Let κ be a regular infinite cardinal. An equational class A of algebras having less than κ -ary operations is binding if and only if M_κ can be fully embedded into it.

4.4. <u>Remark</u>: 4.2 and 4.3 remain true if we suppose that $M_\kappa - \{\underline{1}\}$ can be pseudorealized into A instead of M_κ fully embedded.

Let C be the full subcategory of the category of undirected graphs determined by all connected 3-colourable graphs and the graph $\underline{4}$ and M the full subcategory of C composed of $\underline{1}$, $\underline{2}$ and $\underline{4}$. M is the testing category from [11]. The following result is given in [10] for co-well-powered A and this additional supposition can be left out by 3.7. Similarly, co-well-poweredness can be omitted in Theorem 3 of [10].

4.5. <u>Theorem</u>. Let (A, V) have the property of transfer, let A have and V compress colimits from $\mathcal{C}_*$. Then A is binding if and only if M can be fully embedded into it.

Finally, supposing the existence of many measurable cardinals we shall give an example of a non-binding monadic category containing any small category as a full subcategory. Let $P^- : \mathcal{E}ns \longrightarrow \mathcal{E}ns$ be the contravariant power set functor, i.e. $P^- x = \exp x$ and $P^-(f)(z) = f^{-1}z$ for $f : x \longrightarrow y$ and $z \subseteq y$.

4.6. <u>Lemma</u>. Let (A, V) be a monadic category and V have a right adjoint. Then $(A^{op}, P^- V^{op})$ is monadic.

Proof: Following [5], $(\mathcal{Ens}^{op}, P^-)$ is monadic. Hence P^- has a left adjoint and creates split coequalizers. Further, V^{op} creates all colimits because V is monadic. Putting all these facts together, $P^- V^{op}$ has a left adjoint and creates split coequalizers and therefore is monadic by Beck's precise tripleability theorem.

We recall that (M) denotes the following assumption: There is a cardinal $\mathfrak{m}$ such that every ultrafilter closed under intersections of $\mathfrak{m}$ elements is trivial.

4.7. Example: Let $\mathrm{non}\,(M)$ hold, (A, V) be the category of all algebras with two unary operations, where V is the usual forgetful functor. By 4.6 $(A^{op}, P^- V^{op})$ is monadic. Since A is binding (see [1]), any small category can be fully embedded into A^{op}. By [4] $\mathcal{Ens}^{op}$ cannot be fully embedded into A and therefore A^{op} is not binding.

By 4.5 and 4.7 there is no colimit compressing faithful functor $\mathcal{Ens}^{op} \longrightarrow \mathcal{Ens}$, under $\mathrm{non}\,(M)$.

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