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SUBSEQUENTIAL LIMITS OF FIXED POINT SETS

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Abstract: In this paper a sequence of functions $\{T_n\}$ that map a complete metric space (X, d) into itself and that converge uniformly to $T_0 : X \longrightarrow X$ is considered. If $F(T_n)$ denotes the set of fixed points of T_n and for all $x \notin F(T_n)$ and all n, T_n satisfies

$$d(T_n x, F(T_n)) \le \alpha(d(x, F(T_n)))\, d(x, F(T_n)) + \beta(d(x, F(T_n)))\, d(x, T_n x)$$

where $\alpha : (0, \infty) \longrightarrow [0, 1)$ and $\beta : (0, \infty) \longrightarrow [0, 1)$ are monotonically decreasing functions and $\alpha(d(x, F(T_n))) + 2\beta(d(x, F(T_n))) < 1$, then conditions are given that insure that $F(T_0)$ is nonempty and compact. The work generalizes the result of Bruce Hillam [1] and Diaz and Metcalf [3].

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Introduction. Throughout this paper, (X, d) will denote a complete metric space.

0.1. Definition. Let (X, d) be a metric space. A function $T : X \longrightarrow X$ is said to be strictly contractive if there exists a constant k, $0 \le k < 1$ such that $d(Tx, Ty) \le k d(x, y)$ for all x and y in X.

0.2. Definition. Let (X, d) be a metric space. A function $T : X \to X$ is said to be a contraction if $d(Tx, Ty) < d(x, y)$ for all x and y in X with $x \neq y$.

0.3. Definition. Let (X, d) be a metric space and $\varepsilon > 0$. Then the sets of the form $S_\varepsilon(x) = \{y : d(x,y) < \varepsilon\}$ are called spheres in X. The sphere $S_\varepsilon(x)$ has x for its center, and ε for a radius.

0.4. Definition. Let X be a metric space, and $T : X \to X$ be a function. $F(T)$ is defined to be the set of all fixed points of T.

0.5. Definition. Let (X, d) be a metric space and for $n = 1, 2, 3, 4, \ldots$ let $K_n \in X$ be a sequence of non-empty sets. We define $\mathcal{L}(\{K_n\})$ to be the set of all possible subsequential limit points of all possible sequences $\{k_j\}$ where $k_j \in K_j$, i.e.

$$\mathcal{L}(\{K_n\}) = \{x \in \mathcal{L}(\{k_j\}) \ \forall \{k_j\},\ k_j \in K_j\}$$

In other words, $\mathcal{L}(\{K_n\})$ is the upper limit $L_s K_n$. (See Kuratowski [4], chapt. 2, § 29, III).

0.6. Definition. H is defined to be the family of all functions $\alpha : (0, \infty) \to [0, 1)$ such that α is monotonically decreasing.

Bruce Parks Hillam [1] proved:

Theorem. For $n = 1, 2, \ldots$ let $T_n : X \to X$ be a sequence of functions each of which has at least one fixed point a_n. Let $T_0 : X \to X$ be a function with a unique fixed point a_0 such that for all x in X

(1) $d(T_0 x, a_0) \leq \alpha(d(x, a_0))\, d(x, a_0)$, $\alpha \in H$. Then, if $T_n \to T_0$ uniformly on X, $a_n \to a_0$.

Metcalf and Diaz [3] have considered functions where $d(Tx, F(T)) < d(x, F(T))$, where $F(T)$ is the fixed point set of the function T.

Bruce has shown by an example that if (1) is replaced by

$$d(T_0 x, F(T_0)) \leq \alpha(d(x, F(T_0)))\, d(x, F(T_0))$$

then the sequence of fixed points might not converge but the subsequential limit points are fixed points.

In our present paper we extend a few theorems of Bruce [1] and a theorem of Diaz and Metcalf [3].

If for $n = 1, 2, \ldots$ there is a sequence of functions $T_n : X \longrightarrow X$ such that $F(T_n)$ is nonempty and α, $\beta \in H$ then $\alpha_n(x)$, $\beta_n(x)$ will denote the functions $\alpha_n(x) = \alpha(d(x, F(T_n)))$, $\beta_n(x) = \beta(d(x, F(T_n)))$ and

$$\alpha(T_n x, F(T_n)) \leq \alpha_n(x)\, d(x, F(T_n)) + \beta_n(x)\, d(x, T_n x)$$

will be written instead of

$$d(T_n x, F(T_n)) \leq \alpha(d(x, F(T_n)))\, d(x, F(T_n)) + \\ + \beta(d(x, F(T_n)))\, d(x, T_n x)$$

for each n. The following lemma is due to Bruce [1].

<u>Lemma 1.1</u>. For $n = 1, 2, 3, \ldots$ let $T_n : X \longrightarrow X$ be a sequence of functions such that $F(T_n)$ is nonempty. Let $T_0 : X \longrightarrow X$ be continuous and suppose $T_n \longrightarrow T_0$ uniformly. If $\{a_{i_j}\}$ is a sequence where $a_i \in F(T_i)$ and such that $a_{i_j} \longrightarrow x_0$ then $x_0 \in F(T_0)$ and $L_S F(T_n) \subseteq F(T_0)$.

<u>Lemma 1.2</u>. For $n = 1, 2, \ldots$, let $T_n : X \longrightarrow X$ be a sequence of functions such that $F(T_n)$ is nonempty.

Suppose there are functions α and β in H such that for all $x \in X | F(T_m)$,

(1.2.1) $d(T_m x, F(T_m)) \leq \alpha_m(x) d(x, F(T_m)) +$

$$+ \beta_m(x) d(x, T_m x) \quad \alpha_m(x) + 2\beta_m(x) < 1 .$$

Let $T_0 : X \longrightarrow X$ be a continuous function and suppose $T_m \longrightarrow T_0$ uniformly. Then for every $\varepsilon_0 > 0$ there exists and integer I_0 with the property that for each $a_{I_0} \in F(T_{I_0})$ the following hold .

(i) There exists a convergent sequence $\{a_{i_j}\}$ with

$$a_{I_0} = a_{i_1} \quad \text{and} \quad a_{i_j} \in F(T_{i_j}) \ ;$$

(ii) $d(a_{i_j}, a_{i_k}) < \varepsilon_0$ for all positive integers j, k .

<u>Proof</u>: Let $\varepsilon_0 > 0$ be arbitrary. Set $\varepsilon_1 = \frac{\varepsilon_0}{2}$ and choose ε_1' such that $\frac{\varepsilon_1'}{1-\lambda(\varepsilon_1)} < \varepsilon_1$, $\lambda(\varepsilon_1) = \frac{\alpha(\varepsilon_1)+\beta(\varepsilon_1)}{1-\beta(\varepsilon_1)}$.

Since $T_m \longrightarrow T_0'$ uniformly, there exists a positive integer N_1 such that for all $j, k \geq N_1$, $d(T_k x, T_j x) < \varepsilon_1'$.

Let $I_0 = N_1$, $a_{I_0} \in F(T_{I_0})$ be arbitrary and set $a_{i_1} = a_{I_0}$

<u>Claim 1</u>. For every $k \geq N_1$, $d(a_{i_1}, F(T_k)) < \varepsilon_1 = \frac{\varepsilon_0}{2}$.

If not, then there exists a $k_0 \geq N_1$ such that $d(a_{i_1}, F(T_{k_0})) \geq \varepsilon_1$. But then

$d(a_{i_1}, F(T_{k_0})) \leq d(T_{i_1} a_{i_1}, T_{k_0} a_{i_1}) + d(T_{k_0} a_{i_1}, F(T_{k_0}))$.

Now

$$d(T_{k_0} a_{i_1}, F(T_{k_0}))$$

$$\le \alpha_{k_0}(a_{i_1}) d(a_{i_1}, F(T_{k_0})) + \beta_{k_0}(a_{i_1}) d(a_{i_1}, T_{k_0} a_{i_1})$$

$$\le \alpha_{k_0}(a_{i_1}) d(a_{i_1}, F(T_{k_0})) + \beta_{k_0}(a_{i_1}) d(a_{i_1}, F(T_{k_0}))$$

$$+ \beta_{k_0}(a_{i_1}) d(F(T_{k_0}), T_{k_0}(a_{i_1})) .$$

Or

$$d(T_{k_0} a_{i_1}, F(T_{k_0})) \le \frac{\alpha_{k_0}(a_{i_1}) + \beta_{k_0}(a_{i_1})}{1 - \beta_{k_0}(a_{i_1})} d(a_{i_1}, F(T_{k_0})) .$$

Therefore

$$d(a_{i_1}, F(T_{k_0})) < \varepsilon_1' + \lambda_{k_0}(a_{i_1}) d(a_{i_1}, F(T_{k_0}))$$

where $\lambda_{k_0}(a_{i_1}) = [\alpha_{k_0}(a_{i_1}) + \beta_{k_0}(a_{i_1})] / [1 - \beta_{k_0}(a_{i_1})]$.

This, combined with the fact that α and β are monotone decreasing, implies

$$d(a_{i_1}, F(T_{k_0})) < \frac{\varepsilon_1'}{1 - \lambda_{k_0}(a_{i_1})} \le \frac{\varepsilon_1'}{1 - \lambda(\varepsilon_1)} < \varepsilon_1$$

which is a contradiction.

Let $\varepsilon_2 = \dfrac{\varepsilon_0}{2^2}$ and choose ε_2' such that $[\varepsilon_2'/1 - \lambda(\varepsilon_2)] < \varepsilon_2$ and let $N_2 \ge N_1$ be chosen so that for all $j, k \ge N_2$, $d(T_k x, T_j x) < \varepsilon_2'$. Let $a_{i_2} \in$ $\in F(T_{N_2})$ where $i_2 = N_2$ be chosen such that $d(a_{i_1}, a_{i_2}) < \varepsilon_1$ which is possible by Claim 1.

By an argument similar to Claim 1, $d(a_{i_2}, F(T_k)) < \varepsilon_2$ for all $k \ge N_2$. Suppose that for a finite increasing sequence of integers $\{N_j\}_{j=1}^{n}$ there corresponds a sequence of points $\{a_{i_j}\}_{j=1}^{n}$ such that

(i) $a_{i_j} \in F(T_{N_j})$ where $N_j = i_j$, $j = 1, 2, \dots, n$,

(ii) $$d(a_{i_j}, a_{i_{j+1}}) < \varepsilon_j = \frac{\varepsilon_0}{2^j} ,$$

(iii) $d(a_{i_n}, F(T_k)) < \varepsilon_n = \varepsilon_0 / 2^n$ for all $k \geq N_n$.

Then $N_{n+1}, a_{i_{n+1}}$ are found by setting $\varepsilon_{n+1} = \frac{\varepsilon_0}{2^{n+1}}$, choosing ε'_{n+1} such that $[\varepsilon'_{n+1} / 1 - \lambda(\varepsilon_{n+1})] < \varepsilon_{n+1}$. By the uniform convergence of $\{T_k\}$ there exists a positive integer $N_{n+1} > N_n$ such that for all $k, j \geq N_{n+1}$,

$$d(T_j x, T_k x) < \varepsilon'_{n+1} .$$

Let $i_{n+1} = N_{n+1}$. By (iii) there is an $a_{i_{n+1}}$ in $F(T_{i_{n+1}})$ such that $d(a_{i_n}, a_{i_{n+1}}) < \varepsilon_n = \frac{\varepsilon_0}{2^n}$. Also for all $j \geq N_{n+1}$, $d(a_{i_{n+1}}, F(T_j)) < \varepsilon_{n+1}$. We continue the above procedure and let $\{a_{i_j}\}$ denote the resulting subsequence.

Claim 2. $\{a_{i_j}\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Let N denote the positive integer such that $(\varepsilon_0 / 2^{N+1}) < \varepsilon$. Thus for all $k, j \geq N$,

$$d(a_{i_j}, a_{i_k}) \leq \sum_{t=0}^{k-j-1} d(a_{i_{j+1}}, a_{i_{j+t+1}})$$
$$< \sum_{t=0}^{k-j} (\varepsilon_0 / 2^{j+t}) \leq (\varepsilon_0 / 2^{N+1}) < \varepsilon .$$

Thus $\{a_{i_j}\}$ is a Cauchy. So Lemma 1.2 follows. Combining Lemma 1.1 and Lemma 1.2 the following fixed point theorem is obtained.

<u>Theorem 1.3</u>. For $n = 1, 2, 3, \ldots$, let $T_n : X \to X$ be a sequence of functions such that $F(T_n)$ is nonempty. Suppose there are α and β in H such that for all $x \in$ $\in X - F(T_n)$ (1.2.1) holds. Let $T_0 : X \to X$ be a continuous function and suppose $T_n \to T_0$ uniformly, then $L_S F(T_n)$ is nonempty. Furthermore, $L_S F(T_n) = F(T_0)$ and $F(T_0) = \lim \{F(T_n)\}$.

<u>Proof</u>: By Lemma 1.2, there exists at least one Cauchy subsequence a_{i_j} and since (X, d) is a complete metric space, $\{a_{i_j}\}$ converges to some element of X say u_0. By Lemma 1.1, $u_0 \in F(T_0)$ and $L_S F(T_n) \subseteq F(T_0)$. To show that $F(T_0) = L_S F(T_n)$ it suffices to show that for every $\varepsilon > 0$ and for arbitrary but fixed $a_0 \in F(T_0)$, $\exists$ a positive integer N such that for all $k \geq N$, $d(a_0, F(T_k)) < \varepsilon$. Let ε' be so chosen that

$$\frac{\varepsilon'}{1 - \lambda(\varepsilon)} < \varepsilon \,, \quad \lambda(\varepsilon) = \frac{\alpha(\varepsilon) + \beta(\varepsilon)}{1 - \beta(\varepsilon)} \,.$$

By the uniform convergence of $\{T_n\}$ there is a positive integer N' such that $d(T_k x, T_0 x) < \varepsilon'$ for all $k \geq N'$.

<u>Claim</u>. For all $k \geq N'$, $d(a_0, F(T_k)) < \varepsilon$.

If not, then there is a $j \geq N'$ such that $d(a_0, F(T_j)) \geq \varepsilon$. But then

$$\begin{aligned} & d(a_0, F(T_j)) \\ & \leq d(T_0 a_0, T_j a_0) + d(T_j a_0, F(T_j)) \\ & < \varepsilon' + \lambda_j(a_0)\, d(a_0, F(T_j)) \,. \end{aligned}$$

Or

$$d(a_0, F(T_j)) \leq [\varepsilon'/1 - \lambda_j(a_0)] \,.$$

But α, β are monotone decreasing, so the above implies, by

the choice of ε', $d(a_0, F(T_j)) \leq \dfrac{\varepsilon'}{1-\lambda(\varepsilon)} < \varepsilon$, which is a contradiction. Therefore $F(T_0) \subseteq L_s F(T_m)$. Finally, $F(T_0)$ is the limit of $\{F(T_m)\}$. Indeed, as $\forall \varepsilon > 0$, $\exists N\ \forall k \geq N$, $d(a_0, F(T_k)) < \varepsilon$ it follows $\lim_k d(a_0, F(T_k)) = 0$, i.e. $a_0 \in L_i F(T_k)$.
As $L_s F(T_k) \subseteq F(T_0)$, we have

$$L_s F(T_k) \leq F(T_0) \subseteq L_i F(T_k)) \subseteq L_s F(T_k) \quad ,$$

i.e. $\quad F(T_0) = L_i F(T_k) = L_s F(T_k)$,

so that $F(T_0) = LF(T_k)$.

(For notation L , L_i see Kuratowski [4].)

For the special case that for every integer m , $F(T_m) = \{a_m\}$ and $\alpha, \beta \in H$ are defined to be $\alpha(t) = k_1$, $\beta(t) = k_2$ such that $k_1 + 2k_2 < 1$, T_0 need not be continuous, which is the import of the following theorem.

__Theorem 1.4.__ For $m = 1, 2, 3, \ldots$, let $T_m : X \longrightarrow X$ be a sequence of functions such that $F(T_m) = \{a_m\}$. Suppose there exist strictly positive k_1 and k_2 with $k_1 + 2k_2 < 1$ such that for all $x \in X - \{a_m\}$ and for all m

$$(1.4.1) \qquad d(T_m x, a_m) \leq k_1 d(x, a_m) + k_2 d(x, T_m x) \ .$$

Then if $T_0 : X \longrightarrow X$ is a function such that $T_m \longrightarrow T_0$ uniformly, then $F(T_0)$ is nonempty.

__Proof__: Let $\varepsilon > 0$ be arbitrary. Since $T_m \longrightarrow T_0$ uniformly, there is a positive integer N such that for all $j, m \geq N$, we have

$$d(T_j x,\ T_n x) < (1-k^1)\frac{\varepsilon}{2},\ k^1 = (k_1+k_2)/(1-k_2) .$$

Let $x_0 \in X$ be such that $d(x_0, a_n) < \left(\frac{1-k^1}{4k^1}\right)\varepsilon$.

Then

$$d(a_n, a_j)$$

$$\leq d(T_n x_0, a_n) + d(T_n x_0, T_j x_0) + d(T_j x_0, a_j)$$

$$\leq k^1 d(x_0, a_n) + d(T_n x_0, T_j x_0) + k^1 d(x_0, a_j)$$

$$\leq 2k^1 d(x_0, a_n) + k^1 d(a_n, a_j) + d(T_n x_0, T_j x_0) .$$

Hence

$$d(a_n, a_j) \leq \frac{2k^1}{1-k^1}, d(x_0, a_n) + \frac{1}{1-k^1} d(T_n x_0, T_j x_0) < \varepsilon .$$

Thus $\{a_n\}$ is Cauchy.

Since (X, d) is complete, there exists an $a_0 \in X$ such that $\lim\limits_n a_n = a_0$.

Claim. $T_0 a_0 = a_0$. Let $\varepsilon > 0$ be arbitrary and let N'' be a positive integer such that for all $j \geq N''$, $d(a_j, a_0) < \frac{\varepsilon}{3}$ and for all x, $d(T_j x, T_0 x) < \frac{\varepsilon}{3}$.

Then

$$\begin{aligned} d(a_0, T_0 a_0) &\leq d(a_0, a_j) + d(a_j, T_j a_0) + d(T_j a_0, T_0 a_0) \\ &< \frac{\varepsilon}{3} + k' d(a_j, a_0) + \frac{\varepsilon}{3} < \varepsilon , \end{aligned}$$

which implies $a_0 = T_0 a_0$. Thus $a_0 \in F(T_0)$.

In Theorems 1.3 and 1.4 conditions were given that insured that the limit function T_0 has at least one fixed point

Theorem 1.5 below gives conditions that insure that $F(T_0)$ is compact.

Theorem 1.5. For $n = 1, 2, 3, \ldots$, let $T_n : X \to X$ be a sequence of functions such that $F(T_n)$ is nonempty and compact. Suppose there are α, β in H such that for all n and for all $x \in X - F(T_n)$

(1.5.1) $$\alpha(T_n x, F(T_n))$$

$$\leq \alpha_n(x) d(x, F(T_n)) + \beta_n(x) d(x, T_n x), \alpha_n(x) + 2\beta_n(x) < 1 .$$

Let $T_0 : X \to X$ be a continuous function and suppose that $T_n \to T_0$ uniformly. Then $F(T_0)$ is nonempty and compact.

Proof: By Theorem 1.3, $F(T_0)$ is nonempty, thus it is sufficient to show that $F(T_0)$ is compact. Now, a set in a metric space is compact if and only if it is both complete in itself and totally bounded. Clearly, since T_0 is continous, $F(T_0)$ is complete in itself.

Let $\{a_n\} \subseteq F(T_0)$ be a Cauchy sequence with u_0 as its limit. Thus $u_0 = \lim_n a_n = \lim T_0 a_n = T_0 u_0$ i.e. $u_0 \in F(T_0)$. We wish to show now that $F(T_0)$ is totally bounded. So let $\varepsilon > 0$ be arbitrary. Let ε' be chosen such that $[\varepsilon'/1 - \lambda(\varepsilon - \gamma)] < \varepsilon/\gamma$. By the uniform convergence of the $\{T_n\}$, there exists a positive integer N such that for all $k \geq N$, $d(T_k x, T_0 x) < \varepsilon'$.

Claim 1. For all $a_0 \in F(T_0)$, $d(a_0, F(T_k)) < \varepsilon/\gamma$ for all $k \geq N$. If not, then there exists a $k \geq N$ and an $a_0 \in F(T_0)$ such that $d(a_0, F(T_k)) \geq \varepsilon/\gamma$. Thus

$$d(a_0, F(T_k)) \leq d(T_k a_0, T_0 a_0) + d(T_k a_0, F(T_k)) <$$

$< \varepsilon' + \lambda_k(a_0) d(a_0, F(T_k)), \lambda_k(a_0) = \dfrac{\alpha_k(a_0) + \beta_k(a_0)}{1-\beta_k(a_0)}$

which implies that

$$d(a_0, F(T_k)) < \frac{\varepsilon'}{1-\lambda_k(a_0)}$$

But α_k and β_k are monotone decreasing, this coupled with the choice of ε' gives

$$d(a_0, F(T_k)) < \frac{\varepsilon'}{1-\lambda(\varepsilon/7)} < \varepsilon/7$$

which is a contradiction.

Now from Claim 1 there follows at once:

If S is an $\varepsilon/7$ net for $F(T_k)$, then S is an $2\varepsilon/7$ net for $F(T_0)$ so that $F(T_0)$ is totally bounded. This completes the proof.

<u>Theorem 1.6.</u> Let $T_n : X \to X$ be a sequence of mappings with fixed point a_n for each $n = 1, 2, \ldots$ and let $T_0 : X \to X$ be a strict contraction mapping with fixed point a_0. If the sequence $\{T_n\}$ converges uniformly to T_0 and if a subsequence $\{a_{i_j}\}$ of $\{a_i\}$ converges to a point $x_0 \in X$ then $x_0 = a_0$.

<u>Proof</u>: Let $\varepsilon > 0$. There is a positive integer N such that $j \geq N$ implies $d(a_{i_j}, x_0) < \varepsilon/2$. Therefore

$$d(a_{i_j}, T_0 x_0) = d(T_{i_j} a_{i_j}, T_0 a_{i_j}) + d(T_0 a_{i_j}, T_0 x_0) < \varepsilon$$

for all $j \geq N$.

Thus $\{a_{i_j}\}$ converges to $T_0 x_0$. Thus $x_0 = T_0 x_0$ and since the fixed point a_0 of T_0 is unique, $x_0 = a_0$.

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