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# A REMARK ON A FACTORIZATION THEOREM

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Abstract: A proof is given of a factorization theorem in Banach algebra modules based on the induction theorem.

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Recently, V. Pták obtained a theorem of the closed graph type, the so called induction theorem [3],[4],[5] which gives an abstract description of results based on successive approximations, among others, the factorization theorem in Banach algebra modules. It is the purpose of this remark to strengthen the main result of [3]. Similar results have been obtained also by other authors (see e.g. [1],[2]); however it seems interesting to give a proof based on the induction theorem which puts into evidence the essence of the approximation process; the induction theorem yields the result immediately.

First of all, we recall some formulations and principles following [5]. Given a positive number $r$ and a set $M$ in a metric space $(E, d)$, we denote by $U(M, r) = \{y \in E;\ d(y, M) < r\}$. Let $\{A(r)\}_{r \in (0,t)}$ be a family of subsets of $E$. We define the limit $A(0)$ as follows:

$$A(0) = \bigcap_{0<s\leq t} \left( \bigcup_{r\leq s} A(r) \right)^{-} .$$

The following principle is called

The Induction Theorem. Let $(E, d)$ be a complete metric space, let $\omega$ be a non-negative function defined on the interval $T = \langle 0, t\rangle$ transforming $T$ into itself and such that

$$\sigma(r) = r + w(r) + w(w(r)) + w(w(w(r))) + \dots$$

is finite for each $r \in \langle 0, t\rangle$. Let $\{Z(r)\}_{r\in\langle 0,t\rangle}$ be a family of subsets of $E$.

If $Z(r) \subset U(Z(w(r)), r)$ for each $r \in \langle 0, t\rangle$ then

$$Z(r) \subset U(Z(0), \sigma(r))$$

for each $r \in \langle 0, t\rangle$.

Definition. Let $A$ be a Banach algebra and $F$ be a Banach space which is an $A$-module. We shall say that $F$ possesses an approximate unit of order $\beta$ if, for each $\varepsilon > 0$, each $x \in F$ and $a \in A$ there exists an $e \in A$, $|e| \leq \beta$ such that $|ex - x| < \varepsilon$, $|ea - a| < \varepsilon$.

Theorem. Let $A$ be a Banach algebra and $F$ be a Banach space which is an $A$-module. Let $B$ be the unital Banach algebra deduced from $A$. Suppose that $F$ possesses an approximate unit of order $\beta$.

Then, for each $y \in F$ and every $\varepsilon > 0$, there exist elements $z \in F$ and $a \in A$ such that

$$az = y$$

with $|a| \leq \beta$, $z \in (By)^{-}$, $|z - y| < \varepsilon$.

Proof. Denote by $G(B)$ the set of all invertible elements of $B$. Let $\varepsilon > 0$ be given, let $\alpha$ be a positive number, $\alpha < (\beta + 1)^{-1}$. Consider the complete metric space $A \times F$ with a distance function given by the norm $|(a, x)| = \max(|a|, (\alpha\varepsilon)^{-1}|x|)$. We define, for each $\varkappa \in (0, 1\rangle$, the set $Z(\varkappa) \subset A \times F$ as follows:

$$Z(\varkappa) = \{(a, z);\ |a| \leq \beta(1 - \varkappa),\ (a + \sigma) \in G(B)$$

for some $|\sigma| \leq \varkappa$ and $z = (a + \sigma)^{-1} y\}$.

We intend to show that $Z(\varkappa) \subset U(Z((1 - \alpha)\varkappa, \varkappa)$ for each $\varkappa \in (0, 1\rangle$. In other words, for each $(a, z) \in Z(\varkappa)$, we shall find an $e \in A$, $|e| \leq \beta$ so that

$a' + \sigma' \in G(B)$ with $a' = a + \alpha\sigma e$, $\sigma' = (1 - \alpha)\sigma$ and

$$|(a' + \sigma')^{-1} y - (a + \sigma)^{-1} y| < \varkappa\alpha\varepsilon .$$

Set $b = \alpha e + (1 - \alpha)$. Since $b - 1 = \alpha|e - 1| < 1$ we have $b \in G(B)$ and $b^{-1} = \sum_{n=0}^{\infty}(1 - b)^n = \sum_{n=0}^{\infty}(\alpha(1 - e))^n$. We have, for each $c \in A$,

$$|(b^{-1} - 1)c| \leq \alpha \cdot |(e - 1)c| \cdot (1 - \alpha(\beta + 1))^{-1} .$$

Further, $a' + \sigma' = a + \sigma b = b(b^{-1}a + \sigma)$. The element $a' + \sigma' \in G(B)$ if and only if $b^{-1}a + \sigma \in G(B)$. But we have $b^{-1}a + \sigma - (a + \sigma) = (b^{-1} - 1)a$. Since the set $G(B)$ is open and the mapping $a \longrightarrow a^{-1}$ is continuous in $G(B)$ it follows that we can find $e \in A$ so that $b^{-1}a + \sigma \in G(B)$ and $|(b^{-1}a + \sigma)^{-1} - (a + \sigma)^{-1}| < (2|b^{-1}y|)^{-1}\varkappa\alpha\varepsilon$. Moreover, we can choose $e$ so that $|(b^{-1} - 1)y| < \varkappa\alpha\varepsilon(2|(a + \sigma)^{-1}|)^{-1}$ as well. We have then

$$|(a' + \sigma')^{-1}y - (a + \sigma)^{-1}y| = |(b^{-1}a + \sigma)^{-1}b^{-1}y - (a + \sigma)^{-1}y| \leq$$
$$\leq |(b^{-1}a + \sigma)^{-1}b^{-1}y - (a + \sigma)^{-1}b^{-1}y| + |(a + \sigma)^{-1}b^{-1}y -$$
$$- (a + \sigma)^{-1}y| \leq |(b^{-1}a + \sigma)^{-1} - (a + \sigma)^{-1}||b^{-1}y| +$$
$$+ |(a + \sigma)^{-1}| \cdot |(b^{-1} - 1)y| < \varkappa\alpha\varepsilon .$$

Finally, the element $(0, y)$ belongs to $Z(1)$ and by the induction theorem $(0, y) \in Z(1) \subset U(Z(0), \sigma(1))$, i.e. $|(0, y) - (a, x)| < \sigma(1) = \frac{1}{\alpha}$ for some $(a, x) \in Z(0)$. It follows that $|a| \leq \beta$, $x \in (By)^{-1}$ and $(\alpha e)^{-1} |y - x| < \alpha^{-1}$ and the proof is complete.

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