COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

FREE ALGEBRAS AND AUTOMATA REALIZATIONS IN THE LANGUAGE OF CATEGORIES

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Abstract: Given a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ the category of F-algebras is formed as a generalization of universal algebras. The paper exhibits a construction of free F-algebras and a discussion of its convergence. These results are applied to realizations of behaviours by automata in categories, as defined by Arbib and Manes. We solve their problem: when do minimal realizations exist. A necessary and sufficient condition (under additional assumptions) is that F preserves co-meets of quotient objects (= pushouts of epimorphisms). A stronger result is obtained for normal functors.

Key words: Functor-algebra, free algebra, automata in a category, minimal realizations, co-meet of quotient objects.

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Prior to this paper, V. Trnková characterized functors from sets to sets with minimal realizations (private communication); the present results are independent of hers. I am very much indebted to V. Koubek, J. Reiterman and V. Trnková for valuable discussions on this subject.

I. Free functor-algebras

Given an endofunctor $F: \mathcal{K} \longrightarrow \mathcal{K}$ denote by $\mathcal{K}(F)$ the category of F-algebras (i.e. pairs (T, ω) where T is an object of $\mathcal{K}$ and $\omega: FT \longrightarrow T$) and homomorphisms

$f:(T,\omega)\longrightarrow(T',\omega')$ which are $\mathcal{K}$-morphisms $f: T\longrightarrow T'$ for which the following diagram commutes:

$$\begin{array}{ccc} FT & \xrightarrow{\omega} & T \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FT' & \xrightarrow{\omega'} & T' \end{array}$$

Notice that 1) the category of universal algebras of type $\Delta=\{M_i\}_{i\in I}$ (where M_i are cardinals, possibly infinite, considered as sets) is just $\mathcal{Set}(F)$ where F is the sum of hom-functors, $F=\bigvee_{i\in I} \mathcal{Hom}(M_i,-)$. 2) If F is a monad then the category of monad-algebras is a full subcategory of $\mathcal{K}(F)$. 3) Generalized algebraic categories $A(F,G)$, where F and G are set functors, represent another generalization of the categories of universal algebras (see [1 - 3, 5, 6]) but in case G is the identical functor we have $A(F,G)=\mathcal{Set}(F)$.

The notion of free algebras can be transferred to functor-algebras as follows: let A be an object of $\mathcal{K}$ and let (A^*,φ^A) be an algebra. (A^*,φ^A) is free over A if there exists a morphism $\rho: A\longrightarrow A^*$ such that for each F-algebra (T,ω) and each morphism $f: A\longrightarrow T$ there exists a unique homomorphism $f^*_\omega:(A^*,\varphi^A)\longrightarrow(T,\omega)$ for which $f^*_\omega\rho=f$. (Thus free algebras are just universal arrows of the natural forgetful functor from $\mathcal{K}(F)$ to $\mathcal{K}$.) Free algebras may be obtained by the following algorithm or, more generally, by the following transfinite construction. In what follows, we assume that a cocomplete category $\mathcal{K}$ and a functor $F:\mathcal{K}\longrightarrow\mathcal{K}$ are given.

__Free algebra algorithm__. Given an object $A \in \mathcal{K}^{\sigma}$ put $W_0 = A$; $W_1 = A \vee FA$; $W_2 = A \vee F(A \vee FA), \ldots, W_{n+1} = A \vee FW_n$. Denote $\varphi_n : FW_n \longrightarrow W_{n+1}$ and $h : A \longrightarrow W_1$ the canonical maps.

The algorithm is said to converge if for the colimit A^* ; $t_n : W_n \longrightarrow A^*$ of the diagram

$$W_0 \xrightarrow{h} W_1 \xrightarrow{1 \vee Fh} W_2 \xrightarrow{1 \vee F(1 \vee Fh)} W_3 \ \ldots$$

there exists $\varphi : FA^* \longrightarrow A^*$ with $\varphi F t_n = t_{n+1} \varphi_n$, $n = 0, 1, 2, \ldots$. In that case (A^*, φ) is a free algebra over A with respect to $t_0 : A \longrightarrow A^*$, as will be seen later.

__Definition__. A functor F is said to preserve unions of sequences if it preserves the colimit of any diagram $U_0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow \ldots$ of subobjects.

__Proposition__. If F preserves unions of sequences and $hom(FA, A) \neq \emptyset$ then the free algebra algorithm converges for A . If moreover F preserves countable sums then $A^* = A \vee \bigvee_{n=1}^{\infty} F^n A$.

__Proof__. It is easy, notice that $hom(FA, A) \neq \emptyset$ implies that h is a coretraction and so h, ${}_A \vee Fn, \ldots$ are subobjects.

__Note__. Since each finite hom-functor from sets to sets clearly preserves unions of sequences, the above algorithm directly generalizes the construction of free finitary universal algebras.

<u>Free algebra construction</u>. Given an object A define by transfinite induction objects W_i and morphisms $s_{i,j}$: $: W_j \longrightarrow W_i$ ($j \le i$ are arbitrary ordinals) such that for any ordinal γ a diagram $D_\gamma = (\{W_i\}_{i<\gamma}, \{s_{i,j}\}_{j\le i<\gamma})$ is constituted (i.e., $s_{i,i} = 1_{W_i}$ and $s_{ki} s_{i,j} = s_{kj}$).

$W_0 = A$; $W_1 = A \vee FA$; s_{10} is canonical.

i non-limit: $W_{i+1} = A \vee FW_i$; $s_{i+1,i} = 1_A \vee Fs_{i,i-1}$.

i limit: a) W_i and $s_{i,j} : W_j \longrightarrow W_i$, $j < i$, is the colimit of D_i

b) $W_{i+1} = A \vee FW_i$; $s_{i+1,i}$ is defined by: $s_{i+1,i}\, s_{i,0}$ is canonical, $s_{i+1,i} s_{i,j+1} = 1_A \vee FS_{i,j}$.

The construction is said to stop after α steps if $s_{\alpha+1,\alpha}$ is an isomorphism. Then put $A^* = W_\alpha$, $\varphi^A =$ $= (s_{\alpha+1,\alpha})^{-1} m : FA^* \longrightarrow A^*$, where $m : FW_\alpha \longrightarrow W_{\alpha+1}$ is canonical; put $s^A = s_{\alpha,0}$. Denote by $m_i : A \longrightarrow W_{i+1}$ and $m_i : FW_i \longrightarrow W_{i+1}$ the canonical maps.

<u>Proposition</u>: If the free algebra construction stops then (A^*, φ^A) is a free F-algebra over A with respect to s^A .

To prove the proposition we exhibit a construction of the extension of a morphism $f : A \longrightarrow Q$, where (Q, σ) is an algebra, to a homomorphism $f^*_\sigma : (A^*, \varphi^A) \longrightarrow (Q, \sigma)$. Set $f^*_\sigma = f^\alpha$ where $f^i : W_i \longrightarrow Q$ is defined by induction. $f^0 = f$; $f^{i+1} m_i = f$ and $f^{i+1} m_i = \sigma F f^i$; for i limit $f^i s_{ij} = f^j$ for all $i > j$ defines f^i . f^*_σ is a homomorphism since $f^\alpha = f^{\alpha+1} s_{\alpha+1,\alpha}$ and

$f^{\alpha+1} m_\alpha = \sigma F f^\alpha$ and so $\sigma F f^*_\sigma = f^\alpha (s_{\alpha+1,\alpha})^{-1} m_\alpha = f^\alpha \varphi^A$.
The uniqueness follows since given $g : (A^*, \varphi^A) \longrightarrow (Q, \sigma)$ put $g^i = g s_{\alpha,i}$, then $g^0 = f^0$ implies $g^i = f^i$ for all i.

Note. Koubek and Kůrková-Pohlová presented a construction of free algebras in case $\mathcal{K} =$ sets and mappings. Their construction is easily seen to be essentially the same as the one above, in particular as far as the stop is concerned. They prove that the construction stops for A iff there exists a set $B \supset A$ with $card\, FB = card\, B$. Moreover, if the construction does not stop then free algebras do not exist. We generalize the last result.

Definition. A category $\mathcal{K}$ is said to fulfil the coretract chain condition if it is coretract-locally small and for each well-ordered diagram D of coretractions (D consists of coretractions $p_{i,j} : W_j \longrightarrow W_i$, $j \le i$ are ordinals less than γ) the following holds: if U and $q_i : W_i \longrightarrow U$ is the colimit of D then for each co-bound of coretractions U', $q'_i : W_i \longrightarrow U'$ the unique morphism $q : U \longrightarrow U'$ with $q q_i = q'_i$ is also a coretraction.

Examples. The following categories clearly fulfil the coretract chain condition: 1) sets and mappings, 2) vector spaces (over any field) and linear mappings, 3) sets and relations.

Theorem. If $\mathcal{K}$ fulfils the coretract chain condition and $hom(FA, A) \neq \emptyset$ then there exists a free F-algebra over A iff the free algebra construction stops for A.

Proof. If $hom(FA, A) \neq \emptyset$ then $s_{1,0}^A$ is a coretraction and it follows from the coretract chain condition that all $s_{i,j}^A$ are coretractions. Denote $t_{i,j}: W_i \longrightarrow W_j$ such morphisms that $t_{i,j}\, s_{i,j} = 1_{W_i}$. Assume that (B, ψ) is a free algebra over A with respect to $d_0: A \to B$. To prove that the free algebra construction stops we shall find coretractions $d_i: W_i \longrightarrow B$ with $d_i s_{i,j} = d_j$. Since $\mathcal{K}$ is coretract-locally small there exist $j < i$ such that $s_{i,j}$ is an isomorphism; then so is $s_{j+1,j}$.

Set $d_{i+1}\, m_i = d_0$ and $d_{i+1}\, m_i = \psi F d_i$ and for i limit define d_i by $d_i s_{i,j} = d_j$ for $j < i$. Clearly $d_i s_{i,j} = d_j$, let us prove that d_i are coretractions. Choose $e: FA \longrightarrow A$ then $(1_A)_e^* d_0 = {}_A$ and so d_0 is a coretraction. Put $\sigma = m_i\, F t_{i+1,i}$ then we have the extension of m_i to a homomorphism $f: (B, \psi) \longrightarrow (W_{i+1}, \sigma)$ and a straightforward proof by induction shows $f d_j = s_{i+1,j}$ for all $j \leq i+1$, in particular $f d_{i+1} = 1_{W_{i+1}}$. It follows now from the coretract chain condition that also d_i for i limit are coretractions.

II. Minimal realizations by automata in categories

Following Arbib and Manes we call F an input process if free algebras exist over any generator. Then for fixed objects I, Y of $\mathcal{K}$ we define the category of automata: objects are automata $M = (Q, \sigma, \tau, \beta)$ where (Q, σ) is an F-algebra,

$$\begin{tikzcd} FQ \arrow[r, "\delta"] & Q \arrow[r, "\beta"] & Y \\ & I \arrow[u, "\tau"] & \end{tikzcd} \qquad \begin{tikzcd} & Q \arrow[dr, "\beta"] \arrow[dd, "\sigma"] & \\ I \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & & Y \\ & Q' \arrow[ur, "\beta'"'] & \end{tikzcd} \qquad \begin{tikzcd} FI^* \arrow[r, "\varphi^I"] \arrow[d, "F\tau^*"] & I^* \arrow[d, "\tau^*"] & \\ FQ \arrow[r, "\delta"] & Q \arrow[r, "\beta"] & Y \end{tikzcd}$$

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and $\tau : I \longrightarrow Q$, $\beta : Q \longrightarrow Y$ are morphisms in $\mathcal{K}$; morphisms are simulations $\sigma : \mathcal{M} \longrightarrow \mathcal{M}'$ which means a homomorphism $\sigma : (Q, \delta) \longrightarrow (Q', \delta')$ for which $\tau' = \sigma\tau$, $\beta'\sigma = \beta$. (This generalizes e.g. sequential machines: $\mathcal{K} =$ sets and mappings, $F(-) = - \times \Sigma$ where Σ is the input alphabet, Q are states, β is the output map, δ is the next-state function and τ maps a singleton set I onto the initial state of $\mathcal{M}$.) If the extension $\tau^*_\delta : I^* \longrightarrow Q$ is epi then $\mathcal{M}$ is said to be reachable.

The external behaviour of $\mathcal{M}$ is the morphism $f_M = \beta\tau^*_\delta$. Conversely, a realization of a morphism $f : I^* \longrightarrow Y$ is an automaton $\mathcal{M}$ whose behaviour is f . The realization $\mathcal{M}$ is minimal if 1) $\mathcal{M}$ is reachable, 2) for any other reachable realization $\mathcal{M}'$ there exists a simulation $\sigma : \mathcal{M}' \longrightarrow \mathcal{M}$.

Arbib and Manes asked under which condition minimal realizations exist. For <u>constructive</u> input processes, i.e. functors F for which the free algebra construction always stops, we give an answer in terms of co-meets of quotients.

Let $\{k_t\}_{t \in T}$ be a collection of quotients of an object A , i.e. epis $k_t : A \longrightarrow A_t$. Recall that a <u>co-meet</u>, or a multiple pushout, of the collection is a quotient $k : A \to B$ such that 1) there exist $p_t : A_t \to B$ with $k = p_t k_t$, 2) for each $k' : A \to B'$ and $p'_t : A_t \to B'$ with $k' = p'_t k_t$ there exists $f : B \to B'$ such that $p'_t = f p_t$ for all $t \in T$. In other words, co-meet $k = \bigcap^*_{t \in T} k_t$ is the biggest quotient less than all k_t in the quasiorder $r \leq s$ iff $r = s' s$ for some s'.

A functor F is said to preserve co-meets if for each non-initial object A and each collection of quotients of A their co-meet is preserved by F in the sense of preservation of colimits (shortly: if $F(\bigcap^* k_t) = \bigcap^* F k_t$). Recall that an object A is initial iff for each object X there leads exactly one morphism from A to X. The category $\mathcal{K}$ is called <u>connected</u> if $hom(X,Y) \neq \emptyset$ for arbitrary objects X, Y with one possible exception that Y is initial.

F is said to admit minimal realizations if for each objects I, Y and each $f : I^* \to Y$ there exists a minimal realization of f.

<u>Theorem</u>. Let $\mathcal{K}$ be a cocomplete, connected, co-locally small category. Then a constructive input process F admits minimal realizations if and only if F preserves co-meets.

<u>Proof</u>. I. Sufficiency. Given $f : I^* \to Y$, let $\{(Q_t, \sigma_t, \tau_t, \beta_t) ; t \in T\}$ be the collection of all reachable realizations of f; denote $r_t = (\tau_t)^*_{\sigma_t} : I^* \to Q_t$. Since all r_t are quotients of I^* and $\mathcal{K}$ is co-locally small there is no harm in assuming that T is a set (and

not a proper class). Let $\varkappa = \bigcap^*_{t \in T} \varkappa_t$ with $\varkappa : I^* \longrightarrow Q$ and $p_t : Q_t \longrightarrow Q$, $\varkappa = p_t \varkappa_t$. Since $\beta_t \varkappa_t = f$ there exists a unique $\beta : Q \longrightarrow Y$ with $\beta_t = \beta p_t$ for all t. Since $F\varkappa = \bigcap^* F\varkappa_t$ and $(p_t \delta_t) F\varkappa_t = \varkappa \varphi^I$ there exists a unique $\delta : FQ \longrightarrow Q$ with $p_t \delta_t = \delta F p_t$ (i.e., $p_t : (Q_t, \delta_t) \longrightarrow (Q, \delta)$). It is easy to verify that $(Q, \delta, \varkappa \circ s^I, \beta)$ is a minimal realization of f.

II. Necessity.

A) F preserves epimorphisms with non-initial domain. Let $k : A \longrightarrow B$ be epi, let $p, q : FB \longrightarrow C$ be morphisms with $pFk = qFk$. We shall prove that $p = q$. 1) $C = B$. Recall the construction of extensions - it is clear that if $pFk = qFk$ then $k^*_p = k^*_q : (A^*, \varphi^A) \longrightarrow (B, p)$. Let $M = (Q, \delta, \tau, \beta)$ be a minimal realization of k^*_p. Since clearly k^*_p is epi, we have two reachable realizations of k^*_p: $N^p = (B, p, k, 1_B)$ and $N^q = (B, q, k, 1_B)$. There exists a simulation $\sigma_p : N^p \longrightarrow M$; then $\beta \sigma_p = 1_B$ and σ_p is epi ($\sigma k^*_p = \tau^*_\delta$) and so σ_p is an isomorphism; clearly σ_p^{-1} is a simulation $M \longrightarrow N^p$. Furthermore, there exists a simulation $\sigma_q : N^q \longrightarrow M$; we get a simulation $\sigma = \sigma_p^{-1} \sigma_q : N^q \longrightarrow N^p$. Then $1_B \sigma = {}_B$, thus $\sigma = 1_B$, and $\sigma : (B, q) \longrightarrow (B, p)$. Therefore $p = q$.
2) C is arbitrary. Set $A' = A \vee C$ with $i_A : A \longrightarrow A'$ and $j_A : C \longrightarrow A'$ canonical, analogously B', i_B and j_B. Put $k' = k \vee 1_C : A' \longrightarrow B'$. Since A is non-initial, we may choose $f : C \longrightarrow A$ and put $f' : A' \longrightarrow A$ with $f' i_A = 1_A$ and $f' j_A = f$; $g' : B' \longrightarrow B$ with $g' i_B = 1_B$ and $g' j_B = kf$. To prove that $p = q$ it suffices to show $j_B p F g' = j_B q F g'$: since $hom(A, C) \neq \emptyset$

and $hom(B,A) \neq \emptyset$ we have $hom(B,C) \neq \emptyset$ which implies that there exists h with $h j_B = 1_B$, then $p = h j_B p Fg' F i_B = h j_B q Fg' F i_B = q$. Since k' is epi and clearly $gk = kf'$ we have $j_B p Fg' Fk' = j_B q Fg' Fk'$ which, according to case 1), implies $j_B p Fg' = j_B q Fg'$.

B) F preserves co-meets.

Let $k = \bigcap^*_{t \in T} k_t$ where $k_t : A \longrightarrow X_t$ are epis, $k : A \rightarrow X$ with $k = p_t k_t$. Let $q = \bigcap^*_{t \in T} F k_t$ where $q : FA \longrightarrow Y$ with $q = q_t F k_t$. (A is a non-initial object.) Since Fk and all Fk_t are epis, to prove that f preserves the co-meet it clearly suffices to find a (necessarily unique) $r : FX \longrightarrow Y$ with $r Fk = q$.

Denote $a : X \longrightarrow X \vee Y$ and $b : Y \longrightarrow X \vee Y$ the canonical maps. Define $f^i : W_i^A \longrightarrow (X \vee Y)^* : f^1 = s^{X \vee Y}(k \vee q)$; $f^{i+1} m_i = s^{X \vee Y}$ and $f^{i+1} m_i = \varphi^{X \vee Y}(F f^i)$; for i limit $f^i s_{i,j} = f^j$, $j < i$. Put $f = f^\alpha : A^* \longrightarrow (X \vee Y)^*$ and consider a minimal realization $M = (Q, \delta, q, \beta)$ of f. For each $t \in T$ put $\beta_t = \beta_t^\alpha : X_t^* \rightarrow (X \vee Y)^*$ where $\beta_t^1 = s^{X \vee Y}(p_t \vee q_t)$; $\beta_t^{i+1} m_i = s^{X \vee Y} a p_t$ and $\beta_t^{i+1} m_i = \varphi^{X \vee Y} F \beta_t^i$. Further, put $\tilde{k}_t : A^* \longrightarrow X_t^*$, $\tilde{k}_t = (s^{X_t} k_t)^*_{\varphi^{X_t}}$ and recall $\tilde{k}_t = k_t^\alpha$ where $k_t^0 = s^{X_t} k_t$; $k_t^{i+1} m_i = s^{X_t} k_t$ and $k_t^{i+1} m_i = \varphi^{X_t} F k_t^i$. Since k_t are epis and F preserves epis (recall that A is non-initial) clearly $\tilde{k}_t$ are epis, too; moreover,

$\beta_t \tilde{k}_t = f$ for each $t \in T$. Therefore we have reachable realizations of f : $M_t = (X_t^*, \varphi_t, s^{X_t}, k_t, \beta_t)$ and there exist

$$\begin{tikzcd} FA^* \arrow[r, "\varphi^A"] \arrow[d, "F\tilde{k}_t"'] & A^* \arrow[r, "f"] \arrow[d, "\tilde{k}_t"] & (X \vee Y)^* \\ FX_t^* \arrow[r, "\varphi^{X_t}"] & X_t^* \arrow[ur, "\beta_t"'] & \end{tikzcd} \qquad \begin{tikzcd} FA^* \arrow[r, "\varphi^A"] \arrow[d, "Fg^*"'] & A^* \arrow[r, "f"] \arrow[d, "g^*"] & (X \vee Y)^* \\ FQ \arrow[r, "\delta"] & Q \arrow[ur, "\beta"'] & \end{tikzcd}$$

simulations $\sigma_t : M_t \longrightarrow M$. Then $g = (\sigma_t \, s^{X_t}) k_t$ for each $t \in T$ and so there exists a unique $r_1 : X \longrightarrow Q$ with $g = r_1 k$. To prove that there exists r with $q = r F k$ we find $r_2 : FQ \longrightarrow Y$ with $q = r_2 F g$ (and put $r = r_2 F r_1$).

Since $\beta g^*_\delta = f$ we have $\beta g^*_\delta s_1^A m_0 = f s_1^A m_0 = s^{X \vee Y} b q$ and so $\beta F g = s^{X \vee Y} b q$. Assume $hom(X, Y) \neq \emptyset$ (if contrary, then Y is initial and as $hom(FA, Y) \neq \emptyset$ also FA is initial and so, since $hom(X, A) \neq \emptyset$ implies $hom(FX, FA) \neq \emptyset$, also FX is initial and the case is clear), analogously $hom(F(X \vee Y), X \vee Y) \neq \emptyset$. It follows that then b is a coretraction; choose $c : X \vee Y \to Y$ with $c b = 1_Y$. Further $s^{X \vee Y}$ is a coretraction for $(1_{X \vee Y})^*_\omega \, s^{X \vee Y} = 1_{X \vee Y}$ where $\omega : F(X \vee Y) \longrightarrow (X \vee Y)$ is arbitrary. Then $q = c \, 1^*_{X \vee Y} s^{X \vee Y} b q = c \, 1^*_{X \vee Y} \beta F g$ and we may put $r_2 = c \, 1^*_{X \vee Y} \beta$. This concludes the proof.

Recall that an object 0 is a zero if for each object A there leads just one morphism from 0 to A and just one from A to 0; then for arbitrary objects A, B there leads just one zero morphism (i.e. morphism factorizable through zero) from A to B. Given $f: A \to B$, the kernel (cokernel) of f is the equalizer (coequalizer) of f and the parallel zero morphism.
A category $\mathcal{K}$ with zero is normal if it has kernels and each monomorphism is a kernel of some morphism, dually: conormal. A category both normal and conormal is called exact. A functor $F: \mathcal{K} \to \mathcal{K}$ is normal if it preserves kernels, i.e. $F(\ker f) = \ker Ff$. Analogously for exact functor. The above result on minimal realizations can be strengthened for normal functors: the preservation of co-meets implies that F is an input process.

__Theorem.__ Let $\mathcal{K}$ be an exact, cocomplete, co-locally small category. Then for a normal functor $F: \mathcal{K} \to \mathcal{K}$ the following is equivalent:

1) F is a constructive input process which admits minimal realizations,

2) F preserves co-meets,

3) F is exact and preserves unions of subobjects.

__Proof.__ We proved $1 \to 2$ already.

$2 \to 3$. It suffices to show that F is conormal since then F is exact and it preserves unions, as, given a collection $\{k_i\}_{i \in I}$ of subobjects of an object, clearly $\bigcup k_i = \ker(\bigcap^* \operatorname{coker} k_i)$. Let $f: A \to B$ be arbitrary, there exists an epi-mono factorization $f = me$

(it follows from the fact that $\mathcal{K}$ is cocomplete and co-locally small, see e.g. Herrlich, Strecker 34.1). Then $\mathrm{coker}\, f = \mathrm{coker}\, m$. Since F is normal, Fm is a monomorphism, thus $Fm = \ker \mathrm{coker}\, Fm$, moreover $m = \ker \mathrm{coker}\, m$ and so $\ker(\mathrm{coker}\, Fm) = \ker(F\, \mathrm{coker}\, m)$. Now, F preserves epis (the exactness of $\mathcal{K}$ allows us to stop worrying about the initial object) and so $F\, \mathrm{coker}\, m$ and $\mathrm{coker}\, Fm$ are epis with the same kernel. Therefore $F\, \mathrm{coker}\, m = \mathrm{coker}\, Fm$ and, as Fe is epi, we get $\mathrm{coker}\, Ff = F\, \mathrm{coker}\, f$.

3$\to$1. Since F preserves unions it is clear that in the free algebra construction S_{ω_0, ω_0+1} is epi and so $s_{\omega_0, i}$ is epi for all i: if i is li,it it follows from the preservation of unions and we have $S_{\omega_0, i+1} = 1 \vee FS_{\omega_0, i}$ which is epi if S_{ω_0} is. Since all $S_{\omega_0, i}$ are quotiants of W_{ω_0}, it follows from the co-local smallness that the free algebra construction stops. The preservation of co-meets follows.

Added in proof: Too late I found a paper of M. Barr, whose results are closely related to the current paper. I mean: Coequalizers and Free Triples, Math.Z.166(1970),307-322.

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