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ON MINIMAL REALIZATIONS OF BEHAVIOR MAPS IN CATEGORIAL AUTOMATA THEORY

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Abstract: Input processes $F: \mathcal{S}et \to \mathcal{S}et$, such that each mapping $f: F^{@} I \to Y$ is a behavior map of a minimal machine, are characterized.

Key words: Set functor, free algebra, behavior maps, minimal realization.

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In the present note we characterize all input processes $F: \mathcal{S}et \to \mathcal{S}et$ such that each mapping $f: F^{@} I \to Y$ has a minimal realization, i.e. it is a behavior of a "minimal" machine (see [3]).

The note has three parts. In I, we give a sufficient condition for the existence of minimal realizations in $\mathcal{D}yn(F)$, $F: \mathcal{K} \to \mathcal{K}$ (see [3]). In II, we apply it to the case $\mathcal{K} = \mathcal{S}et$ and solve fully this situation. In III, we give a very simple sufficient condition for the existence of free F-algebra over any finite set and for the existence of minimal realizations of each $f: F^{@} I \to Y$ with I finite.

I.

1. Let $\mathcal{K}$ be a category, $F: \mathcal{K} \to \mathcal{K}$ be a functor. The category $\mathcal{D}yn(F)$ is defined in [3] as follows x): objects (called F-dynamics) are pairs (X, σ) where $X \in obj\,\mathcal{K}$, $\sigma \in \mathcal{K}(FX, X)$; morphisms (called dynamorphisms) $f: (X, \sigma) \to (X', \sigma')$ are those morphisms $f \in \mathcal{K}(X, X')$ which satisfy $\sigma' \circ Ff = f \circ \sigma$. Let $f: X \to Y$ be a morphism of $\mathcal{K}$, $\sigma = (X, \sigma)$ be an F-dynamics. Any pair (g, σ'), where $g: \sigma \to \sigma'$ is a dynamorphism and f factorizes through g, is called an σ __-realization of__ f xx).

Let $(\mathcal{E}, \mathcal{M})$ be an image factorization system for $\mathcal{K}$ (see e.g.[2]). We say that the σ-realization is __reachable__ if $g \in \mathcal{E}$. We say that (g_1, σ_1) is a __minimal__ σ __-realization of__ f if it is a reachable σ-realization of f and for any reachable σ-realization (g_2, σ_2) of f there exists exactly one dynamorphism $h: \sigma_2 \to \sigma_1$ such that $h \circ g_2 = g_1$.

2. Let $\mathcal{E}$ be a class of morphisms of a category $\mathcal{K}$. A diagram $\mathcal{D}: D \to \mathcal{K}$ is called an $\mathcal{E}$-spectrum if

(i) D is a thin category and for each $\sigma, \sigma' \in obj\,D$ there exists $\sigma'' \in obj\,D$ such that $D(\sigma, \sigma'') \neq \emptyset \neq D(\sigma', \sigma'')$.

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x) The categories $\mathcal{D}yn(F)$ are closely related to the generalized algebraic categories $A(F, G)$ considered in [1],[5],[7],[8].
Here, F, G are set-functors (i.e. endofunctors of $\mathcal{S}et$) and if $G = ident$, then $A(F, G) = \mathcal{D}yn(F)$.

xx) This notion is a simple generalization of realization of a behavior map, considered in [3]. Realizations precisely in the sense of [3] are considered in II and III of the present note.

(ii) for each morphism m of D, $\mathcal{D}(m)$ is in $\mathcal{E}$.

We say that a push-out

$$(*) \qquad \begin{array}{ccccc} & & Y & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\searrow} & \\ X & & & & V \\ & \underset{\gamma}{\searrow} & & \underset{\delta}{\nearrow} & \\ & & Z & & \end{array}$$

is an $\mathcal{E}$-push-out if $\alpha, \gamma \in \mathcal{E}$.

Let $F : \mathcal{K} \longrightarrow \mathcal{K}$ be a functor. We say that F <u>preserves $\mathcal{E}$-push-outs</u> (<u>or colimits of $\mathcal{E}$-spectra</u>) if the image of each $\mathcal{E}$-push-out is a push-out (or the image of a colimit of any $\mathcal{E}$-spectrum $\mathcal{D}$ is a colimit of $F \circ \mathcal{D}$).

3. Let $\mathcal{E}$ be a class of morphisms of a category $\mathcal{K}$. We say that $\mathcal{E}$ is <u>factor-admissible</u> if

(a) $\beta, \gamma \in \mathcal{E}$ whenever $(*)$ is an $\mathcal{E}$-push-out;

(b) $\alpha_d \in \mathcal{E}$ for all $d \in obj\, D$, where $\langle W; \{\alpha_d \mid d \in obj\, D\} \rangle = colim\, \mathcal{D}$, $\mathcal{D}$ is an $\mathcal{E}$-spectrum.

We notice, that, for example, the class epi of all epimorphisms of $\mathcal{K}$ is factor-admissible.

4. Let $\mathcal{E}$ be a class of epimorphisms of a category $\mathcal{K}$. We recall that $\mathcal{E}$-factor object of $X \in obj\, \mathcal{K}$ is any pair (q, X'), where $q \in \mathcal{K}(X, X')$, $q \in \mathcal{E}$. $\mathcal{E}$-factor objects (q_1, X_1), (q_2, X_2) of X are isomorphic if there exists an isomorphism $\sigma \in \mathcal{K}(X_1, X_2)$ such that $\sigma \circ q_1 = q_2$. $\mathcal{K}$ is said to be $\mathcal{E}$-co-well-powered if each its object has only a set of non-isomorphic $\mathcal{E}$-factor objects.

5. <u>Theorem</u>. Let $(\mathcal{E}, \mathcal{M})$ be an image factorization system for a category $\mathcal{K}$, $\mathcal{E}$ be factor admissible. Let $\mathcal{K}$ have $\mathcal{E}$-push-outs and colimits of $\mathcal{E}$-spectra and a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ preserves them. If $\mathcal{K}$ is $\mathcal{E}$-co-well-powered, then each morphism $f: X \longrightarrow Y$ of $\mathcal{K}$ has a minimal σ-realization in $\mathcal{D}yn(F)$ for any F-dynamics $\sigma = (X, \delta)$.

<u>Proof</u> is a routine induction and therefore it is omitted.

6. <u>Proposition</u>. Let $\mathcal{K}$ be a category with coproducts, $\mathcal{E}$ be a class of its epimorphisms. Let Ω be the system of all functors $F: \mathcal{K} \longrightarrow \mathcal{K}$ which preserve $\mathcal{E}$-push-outs and colimits of $\mathcal{E}$-spectra. Then Ω is closed under forming coproducts over a set. If, moreover, $\mathcal{K}$ is complete, is factor-admissible and each $\varepsilon \in \mathcal{E}$ is a retraction (i.e. there exists a morphism μ of $\mathcal{K}$ such that $\varepsilon \circ \mu = 1$), then Ω is closed under forming factor-functors.

<u>Proof</u>. Clearly, Ω is closed under forming coproducts over a set. Let $\mathcal{K}$ be complete, $\mathcal{E}$ be factor-admissible and each $\varepsilon \in \mathcal{E}$ is a retraction. Let F be in Ω, $\nu: F \longrightarrow G$ be an epitransformation.

a) We prove that G preserves $\mathcal{E}$-push-outs. Let $(*)$ be an $\mathcal{E}$-push-out, $\bar{\beta}: GY \longrightarrow W$, $\bar{\delta}: GZ \longrightarrow W$ be morphisms such that $\bar{\beta} \circ G\alpha = \bar{\delta} \circ G\gamma$. Then there exists exactly one $\varphi: FV \longrightarrow W$ such that $\bar{\beta} \circ \nu_Y = \varphi \circ F\beta$, $\bar{\delta} \circ \nu_Z = \varphi \circ F\delta$. Now, it is sufficient to show that φ factorizes through ν_V. Find $\mu: V \longrightarrow Y$ such that $\beta \circ \mu = 1_V$. Then

$\varphi = \varphi \circ F\beta \circ F\mu = \bar{\beta} \circ \nu_Y \circ F\mu = \bar{\beta} \circ G\mu \circ \nu_V$.

b) The proof that G preserves colimits of $\mathcal{E}$-spectra is analogous.

7. Examples:

A) $\mathcal{K} = \mathit{Set}$: Set is cocomplete, (epi, mono) is the only image factorization system of Set, epi is factor-admissible and each its element is a retraction.

Lemma: Let M be a finite set. Then $\mathit{Hom}(M,-)$: $\mathit{Set} \to \mathit{Set}$ preserves epi-push-outs and colimits of epi-spectra.

Proof. We sketch the proof for $F \simeq \mathit{Hom}(2,-)$ given by $FX = X \times X$, $Ff = f \times f$.

a) Let $(*)$ be an epi-push-out, $f: FY \to W$, $g: FZ \to W$ be mappings such that $f \circ F\alpha = g \circ F\gamma$. Define $h: FV \to W$ by $h(x) = (f \circ F\alpha)(x)$, where $x \in FX$ is chosen such that $(F(\beta \circ \alpha))(x) = x$. It is sufficient to prove that $(f \circ F\alpha)(x) = (f \circ F\alpha)(\bar{x})$ whenever $(F(\beta \circ \alpha))(x) = (F(\beta \circ \alpha))(\bar{x})$. We have $x = \langle x_1, x_2 \rangle$, $\bar{x} = \langle \bar{x}_1, \bar{x}_2 \rangle$ and the last equation implies $\beta \circ \alpha(x_1) = \beta \circ \alpha(\bar{x}_1)$, $\beta \circ \alpha(x_2) = \beta \circ \alpha(\bar{x}_2)$. Since $(*)$ is a push-out, there exist chains

$$x_1 = t_0^1,\ t_1^1, \dots, t_n^1 = \bar{x}_1 \quad \text{and} \quad x_2 = t_0^2,\ t_1^2, \dots, t_m^2 = \bar{x}_2$$

such that $\alpha(t_i^j) = \alpha(t_{i+1}^j)$ for i odd, $\gamma(t_i^j) = \gamma(t_{i+1}^j)$ for i even, $j = 1, 2$. Consider the chain

$$\langle x_1, x_2 \rangle = \langle t_0^1, x_2 \rangle, \langle t_1^1, x_2 \rangle, \dots, \langle t_n^1, x_2 \rangle = \langle \bar{x}_1, t_0^2 \rangle, \langle \bar{x}_1, t_1^2 \rangle, \dots, \langle \bar{x}_1, \bar{x}_2 \rangle .$$

b) Let $\mathcal{D} : D \to \mathcal{S}et$ be an epi-spectrum, $\langle X; \{\alpha_d \mid d \in obj\, D\}\rangle = colim\, \mathcal{D}$. Then α_n are epi, so $F\alpha_n$ are epi. It is sufficient to prove that for each $x \in F\mathcal{D}(d)$, $x' \in F\mathcal{D}(d')$ such that $(F\alpha_d)(x) = (F\alpha_{d'})(x')$ there exists $c \in obj\, D$ such that $D(d,c) \neq \emptyset \neq D(d',c)$ and $(F\mathcal{D}(\substack{c\\d}))(x) = (F\mathcal{D}(\substack{c\\d'}))(x')$. Since $x = \langle x_1, x_2\rangle, x' = \langle x_1', x_2'\rangle$, we have $\alpha_d(x_i) = \alpha_{d'}(x_i')$. Find $c_i \in obj\, D$ such that $(\mathcal{D}(\substack{c_i\\d}))(x_i) = (\mathcal{D}(\substack{c_i\\d'}))(x_i')$ and choose c such that $D(c_1,c) \neq \emptyset \neq D(c_2,c)$.

Corollary: If F is a factorfunctor of any $\coprod_{a\in A} Hom(M_a, -)$, where A is a set and all M_a are finite sets, then each mapping $f : X \to Y$ has a minimal σ-realization in $\mathcal{D}yn(F)$ with any $\sigma = (X, \delta)$.

B) $\underline{\mathcal{K} = \mathcal{V}ect}$ (i.e. the category of all real vector spaces and all linear mappings). $\mathcal{V}ect$ is cocomplete, $(epi, mono)$ is the only image factorization system for $\mathcal{V}ect$, epi is factor-admissible and each its element is a retraction.

Lemma: If M is a finite dimensional vector space, then $\mathcal{H}om(M,-) : \mathcal{V}ect \to \mathcal{V}ect$ preserves epi-push-outs and colimits of epi-spectra.

The proof is omitted.

Corollary: If F is a factorfunctor of any $\coprod_{a\in A} \mathcal{H}om(M_a, -)$, where A is a set and all M_a are finite-dimensional vector spaces, then each linear mapping $f : X \to Y$ has a minimal σ-realization in $\mathcal{D}yn(F)$

with any $\sigma = (X, \delta)$.

II.

1. Let $F: \mathcal{K} \longrightarrow \mathcal{K}$ be an endofunctor, $T: \mathcal{D}yn(F) \rightarrow \rightarrow \mathcal{K}$ be the forgetful functor, i.e. $T(X,\delta) = X$, $Tf = f$. We recall (see [3]) that F is called an <u>input process</u> if T has a left adjoint. Denote it by $L: \mathcal{K} \longrightarrow \mathcal{D}yn(F)$. Put $F^{@} = T \circ L$, let $\eta : \mathcal{I}dent \longrightarrow F^{@}$ be the transformation given by the adjunction. Denote $LX = (F^{@}X, \ell_X)$. If $f: F^{@}X \longrightarrow Y$ is a morphism of $\mathcal{K}$, then its LX -realization is called <u>realization</u> only (see [3]).

2. All input processes $F: \mathcal{S}et \longrightarrow \mathcal{S}et$ are characterized in [5]. We recall that a set-functor F is an input process if and only if it is not excessive (a set-functor F is excessive iff $card\, FX > card\, X$ for all sets X with $card\, X \geq m$ for some cardinal number m).

3. <u>Theorem</u>. Let F be a set-functor. The following assertions are equivalent.

(1) F preserves epi-push-outs and colimits of epi-spectra.

(2) For each mapping $f: X \longrightarrow Y$ and each F-dynamics $\sigma = (X, \delta)$, there exists a minimal σ-realization of f .

(3) For each infinite set X , each mapping $f: X \rightarrow 2$ and each F-dynamics $\sigma = (X, \delta)$ there exists a minimimal σ-realization of f .

(4) F is an input process and each mapping

$f : F^{@}X \longrightarrow Y$ has a minimal realization.

(5) F is an input process and each mapping $f : F^{@}X \longrightarrow 2$, with X infinite, has a minimal realization.

(6) F is a factor-functor of some $\coprod_{a \in A} Hom(M_a, -)$, where A is a set and all M_a are finite sets.

4. (6) $\Longrightarrow$ (1) follows from I.7, (1) $\Longrightarrow$ (2) from I.5, (2) $\Longrightarrow$ (3) is evident. (6) $\Longrightarrow$ (4) follows from I.5, 6, 7 and [5], because $\coprod_{a \in A} Hom(M_a, -)$ and their factor-functors are not excessive, (4) $\Longrightarrow$ (5) is evident. Thus, we have to prove the implications (3) $\Longrightarrow$ (6) and (5) $\Longrightarrow$ (6). This is the aim of the rest of II .

5. Let $F : \mathcal{S}et \longrightarrow \mathcal{S}et$ be a functor. If X is a set, define

$$X_F = \bigcup_{\substack{f : Y \to X \\ card\, Y < card\, X}} (Ff)(FY) \quad .$$

We recall (see [4]) that a cardinal m is called an <u>unattainable cardinal of</u> F if $X_F \neq \emptyset$, where $card\, X = m$. F is not a factorfunctor of any $\coprod_{a \in A} Hom(M_a, -)$, where A is a set and all M_a are finite sets if and only if F has an infinite unattainable cardinal (it follows from the Yoneda lemma).

6. <u>The proof of non (6) $\Longrightarrow$ non(3):</u> Let $F : \mathcal{S}et \longrightarrow \mathcal{S}et$ be a functor, which is not a factor-functor of any $\coprod_{a \in A} Hom(M_a, -)$, where A is a set and all M_a are finite sets. Let Y be an infinite set such that $Y_F \neq \emptyset$

(i.e. $\mathrm{card}\, Y$ is an unattainable cardinal of F). Put $X = Y \cup \{a\}$, where a is not in Y, $Z = X \times \{0,1\}$ and we suppose $X \cap Z = \emptyset$. Let $v_0, v_1 : Y \to Z$ be mappings given by $v_i(y) = \langle y, i \rangle$, $i = 0, 1$. Let

$$f : Z \to 2$$

be given by $f(\langle a, 1 \rangle) = 1$, $f(z) = 0$ otherwise. Denote by $\mathbb{K}$ the set of all finite subsets of Y. If $K \in \mathbb{K}$, put $Z_K = K \cup [(X-K) \times \{0,1\}]$, $\varphi_K : Z \to Z_K$ is given by $\varphi_K(\langle x, i \rangle) = x$ whenever $x \in K$, $i = 0, 1$, $\varphi_K(z) = z$ otherwise. If $K \subset K'$, denote by $\varphi^K_{K'} : Z_K \to Z_{K'}$ the mapping such that $\varphi_{K'} = \varphi^K_{K'} \circ \varphi_K$. Clearly, f factorizes through each φ_K. If $i = 0, 1$, put $A^i = [Fv_i](Y_F)$, $A^i_K = [F(\varphi_K \circ v_i)](Y_F)$. Thus, if $K \subset K'$, then $A^i_{K'} = [F\varphi^K_{K'}](A^i_K)$. Since $\varphi_K \circ v_0(Y) \cap \varphi_K \circ v_1(Y)$ is finite, $A^0_K \cap A^1_K = \emptyset$.

Put $B^i = \bigcup_{K \in \mathbb{K}} [F\varphi_K]^{-1}(A^i_K)$, $B^i_K = \bigcup_{\substack{K' \in \mathbb{K} \\ K' \supset K}} [F\varphi^K_{K'}]^{-1}(A^i_{K'})$.

Then $B^0 \cap B^1 = \emptyset$, $B^0_K \cap B^1_K = \emptyset$. Let $\sigma = (Z, \sigma)$ be an F-dynamics, defined as follows. $\sigma(z) = \langle a, 1 \rangle$ if $z \in B^1$, $\sigma(z) = \langle a, 0 \rangle$ otherwise. We show that f has not a minimal σ-realization.

a) First, we define $\sigma_K : FZ_K \to Z_K$ such that $\varphi_K : (Z, \sigma) \to (Z_K, \sigma_K)$ is a dynamorphism. It is sufficient to put $\sigma_K(z) = \langle a, 1 \rangle$ if $z \in B^1_K$, $\sigma_K(z) = \langle a, 0 \rangle$ otherwise.

b) Let (t, σ') be a minimal σ-realization of f, $\sigma' = (T, \tau)$. Since t factorizes through each φ_K, it factorizes through the mapping $h : Z \to \{\langle a, 0 \rangle, \langle a, 1 \rangle\} \cup Y$

given by $h(\langle a,i\rangle)=\langle a,i\rangle, h(\langle y,i\rangle)=y$ if $y\in Y$, $i=0,1$. But if $c\in Y_F$, then $c^i=[Fv_i](c)\in A^i$ and $(Fh)(c^0)=$ $=[F(h\circ v_0)](c)=[F(h\circ v_1)](c)=[Fh](c^1)$, so $(\tau\circ Ft)(c^0)=$ $=(\tau\circ Ft)(c^1)$. On the other hand, $\delta c^0=\langle a,0\rangle$, $\delta c^1=$ $=\langle a,1\rangle$ and $f(\langle a,0\rangle)\neq f(\langle a,1\rangle)$, so $(t\circ\delta)(c^0)\neq$ $\neq(t\circ\delta)(c^1)$, which is impossible.

7. <u>The proof of non(6) ⟹ non(5)</u>: Let $Y, a, X, Z, \sigma=(Z,\delta)$, f have the same meaning as in 6. Let us suppose that F is an input process, let

$$x : F^{@}Z \longrightarrow Z$$

be the mapping such that $x\circ\eta_Z = ident_Z$ and $x:(F^{@}Z,\ell_2)\longrightarrow\sigma$ is a dynamorphism. Put

$$g : F^{@}Z \xrightarrow{x} Z \xrightarrow{f} 2 .$$

Then, g has not a minimal realization in $\mathcal{D}yn\,F$, the proof is the same as in 6.

III.

1. Let $F : \mathcal{S}et \longrightarrow \mathcal{S}et$ be a functor. If F is an input process, then for each set X, there exists a free F-algebra $(F^{@}X, \ell_X)$ over X (i.e. X is embedded in $F^{@}X$ by the mapping $\eta_X : X\longrightarrow F^{@}X$ such that for each mapping $f : X\longrightarrow Y$ and each F-dynamics (Y,δ) there exists exactly one dynamorphism $g:(F^{@}X, \ell_X)\longrightarrow$ $\longrightarrow(Y,\delta)$ such that $g\circ\eta_X = f$). But free F-algebras may exist over <u>some</u> sets X although F is not an in-

put process.

2. Theorem. Let $F: Set \longrightarrow Set$ be a functor such that $card\, F\aleph_0 \leq \aleph_0$ x). Then for each non-empty finite or countable set X there exists a free F-algebra $(F^{@}X, \ell_X)$ over X and each mapping $f: F^{@}X \longrightarrow Y$ has a minimal realization in $\mathcal{D}yn(F)$.

Proof. Since $card\, F\aleph_0 \leq \aleph_0$, $\aleph_0$ is not an unattainable cardinal of F (see [4]). Thus,

$F\aleph_0 = \bigcup_{m=1}^{\infty} (Fi_m)(FA_m)$ whenever $\aleph_0 = \bigcup_{m=1}^{\infty} A_m$,

$A_m \subset A_{m+1}$ and $i_m: A_m \longrightarrow \aleph_0$ is the inclusion. This implies that the algorithm for the construction of a free F-algebra over a set X, described in [5], stops at ω_0 whenever $X \neq \emptyset$ and $card\, X \leq \aleph_0$. Hence, $(F^{@}X, \ell_X)$ exists and $card\, F^{@}X \leq \aleph_0$. Now, we define a subfunctor G of F by $G(Y) = \bigcup_{f: K \to Y,\ K \text{ finite}} (Ff)(FK)$, Gf is a domain-range restriction of Ff. Then $GX = FX$, $G^{@}X = F^{@}X$ whenever $card\, X \leq \aleph_0$. Since G has no infinite unattainable cardinal, it is a factor-functor of some $\coprod_{a \in A} Hom(M_a, -)$, A is a set, M_a are finite. Thus, if $card\, X \leq \aleph_0$, each mapping $f: G^{@}X = F^{@}X \longrightarrow Y$ has a minimal realization in $\mathcal{D}yn(G)$, so in $\mathcal{D}yn(F)$.

R e f e r e n c e s

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x) As usual, $\aleph_0 = \omega_0$ is the set of all smaller ordinal numbers.

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