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ON THE GEOMETRIC CHARACTERIZATION OF DIFFERENTIABILITY I.

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Abstract: In the first part of the paper, the geometric characterization of differentiability in Banach spaces in terms of tangent flats (planes) is given. In the second one, the possibility of such characterization in terms of tangent cones [4] is discussed answering a problem of T.M. Flett [4].

Key words: Banach space, derivative of mapping, tangent flat (plane), tangent cone.

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The differentials of mappings are usually introduced in an analytic (increment) manner, the typical example of which being the definition in the sense of Fréchet, but differentiability can be characterized also in another way: geometrically, i.e., using the notion of a tangent as in the classical analysis. Unfortunately, the simple transposition of a classical notion of a tangent into the spaces of more dimensions or into infinitely dimensional spaces, meets various difficulties. This and other related problems were studied by many authors, for example in [1] - [10]. There are two main directions in approaching the problem of the geometric characterization of differentiability; in the first, the notion of a tangent plane (see [6]) is used, the second is based on the notion of a tangent cone (see [4] and [5]). In both these directions, the characterization was stated in case of finitely dimensional spaces; the aim of our paper is to give such characterizations for infinitely dimensional spaces, too.

The first part of our paper is devoted to generalizing the characterization stated by Roetman [6] to the infinitely dimensional case; the possibility of such generalization was indicated already in [6]. In the second part, we deal with the notion of a tangent cone in the sense of Flett [4]. Flett put the problem ([4], see also [5]) of the characterization of differentiability in infinitely dimensional spaces in terms of tangent cones defined in [4]; we shall show by an example that such a characterization, even under very strong restrictions, is not possible. This problem is investigated also in our paper [11], where we define a slight modified notion of tangent cone and prove the required characterization in terms of the cones in question.

1. Characterization in terms of tangent flats

(1.1) First we recall the main result of Roetman [6]. Let $A \subset \mathbb{R}^m$ be a set with a non-empty interior, $F: A \longrightarrow \mathbb{R}^n$ a mapping and denote $\mathcal{G}(F) = \{(x, y): x \in A, y \in \mathbb{R}^n, y = F(x)\} \subset \mathbb{R}^m \times \mathbb{R}^n$ the graph of F. Consider maximum norms in $\mathbb{R}^m$ and $\mathbb{R}^n$ and the sum norm (i.e. $\|x\|_m + \|y\|_n$) in the product $\mathbb{R}^m \times \mathbb{R}^n$. Let (x_0, y_0) be an interior point of $\mathcal{G}(F)$. A plane Π

(in general, more than 2-dimensional) is said to be the tangent plane to the graph $\mathcal{G}(F)$ at the point (x_0, y_0) if there are $(m+p-1)$-dimensional planes Π_i ($i = 1, \ldots, p$) such that $\Pi = \bigcap_{i=1}^{p} \Pi_i$ and that for arbitrary non-degenerated co-cones $\mathcal{C}'_i(x_0, y_0) \supset \Pi_i$ $(i = 1, \ldots, p)$, an open ball $B(x_0, y_0)$ with the centre at (x_0, y_0) can be chosen so that

$$\mathcal{G}(F) \cap B(x_0, y_0) \subset \bigcap_{i=1}^{p} \mathcal{C}'_i(x_0, y_0) .$$

A co-cone in a space $\mathbb{R}^n$ with a vertex $x_0 \in \mathbb{R}^n$ is defined [6] as a complement in $\mathbb{R}^n$ of the set $\mathcal{C}(x_0) \cup \cup (2x_0 - \mathcal{C}(x_0))$ where $\mathcal{C}(x_0)$ is an open convex cone in $\mathbb{R}^n$ with a vertex at x_0. In these terms, the following theorem holds [6]:

A mapping $F: A \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is differentiable (in Fréchet sense) at an interior point x_0 of A if and only if there is a tangent plane Π to the graph of F at x_0 which is not parallel to the space $\mathbb{R}^p$.

The proof of this theorem is based on the representation of the mapping F by a matrix and on the description of the geometric relations above in the analytic way. Now, following the basic Roetman's ideas, we shall prove an analogical assertion for mappings in Banach spaces.

(1.2) Let Z be a Banach space. A set $\Pi \subset Z$ is said to be a flat (or linear variety) in Z iff it is a translation of some linear subspace of Z; that means $(\Pi - x)$ is a linear subspace of Z for every $x \in \Pi$. A translation of a maximal proper linear subspace of Z is called the hyperplane in Z; if Π is a hyperplane in Z, $x_0 \in \Pi$ and $x_1 \in Z \setminus \Pi$ then Z is the direct sum of $\Pi - x_0$ and $sp(x - x_0)$. Furthermore, if Π is a hyperplane in Z then there is a linear functional $x^*: Z \to \mathbb{R}$ such that $\Pi = \{x : \langle x, x^* \rangle = 0\}$ and on the other hand, the set $\Pi = \{x : \langle x, x^* \rangle = 0\}$ is a hyperplane for every $x^*: Z \to \mathbb{R}$; moreover, Π is closed iff x^* is continuous. See e.g. [12] for these and other properties of hyperplanes used below.

Let $x_0 \in Z$. A set $\mathscr{C}(x_0)$ is said to be the cone in Z with the vertex x_0 iff $\lambda(\mathscr{C}(x_0) - x_0) \subset (\mathscr{C}(x_0) - x_0)$ for every $\lambda > 0$. If $\mathscr{C}(x_0)$ is a convex cone in Z with a vertex x_0 then we call the complement of $\mathscr{C}(x_0) \cup \cup (2x_0 - \mathscr{C}(x_0))$ in Z the co-cone to $\mathscr{C}(x_0)$ and we denote it by $\mathscr{C}'(x_0)$; it is also a cone with a vertex at x_0, but it is not convex.

We shall see later that it is sufficient for the characterization of differentiability to consider a special type of co-cones only. The reason of it lies in the following: If $\mathscr{C}(x_0)$ is a convex cone in Z with a vertex x_0 and if Π is a closed support-hyperplane of $\mathscr{C}(x_0)$ at x_0 such that

$$d\,[S_1(x_0) \cap \mathscr{C}(x_0), \Pi] = \sigma > 0$$

where $S_1(x_0) = \{x : \|x - x_0\| = 1\}$ and $d(A,B) = \inf_{a \in A, b \in B} \|a - b\|$, then the set

$$\{x : x = x_0 + \lambda x', \lambda \geq 0, \|x'\| = 1, d(x_0 + x', \Pi) \leq \tfrac{1}{2}\sigma\}$$

is a subset of the co-cone $\mathscr{C}'(x_0)$. This set is a co-cone, too; moreover, in the case of a finitely dimensional space Z, it is the co-cone to some circular cone with the axis perpendicular to Π ; hence passing to the infinitely dimensional case, we define:

Definition. Let Z be a Banach space, Π a hyperplane in Z, $z_0 \in \Pi$ and $\alpha > 0$. The set

$$\mathcal{C}'_{\Pi,\alpha}(z_0) = \{z : z = z_0 + \lambda z', \lambda \geq 0, \|z'\| = 1, d(z_0 + z', \Pi) \leq \alpha\}$$

is said to be the circular co-cone in Z with vertex z_0 corresponding to the hyperplane Π and the parameter α.

The co-cone $\mathcal{C}'_{\Pi,\alpha}(z_0)$ can be described also in another way which seems to be more suitable for the considerations below. The construction is as follows: Let Π be a closed hyperplane in Z, $z_0 \in \Pi$ and $\alpha > 0$. Choose some $u \in Z \setminus \Pi$, $\|u - z_0\| = 1$ and let $z^*_u \in Z^*$ be such that $\|z^*_u\| = 1$, $\langle u - z_0, z^*_u \rangle = d(u, \Pi)$ and $\langle z - z_0, z^*_u \rangle = 0$ for every $z \in \Pi$; such z^*_u exists due to the Hahn-Banach Theorem. Then

$$(1) \quad \mathcal{C}'_{\Pi,\alpha}(z_0) = \{z : |\langle z - z_0, z^*_u \rangle| \leq \alpha \|z - z_0\|\} .$$

Its validity and the independence of the choice of u and z^*_u follow immediately from the lemma below.

Lemma 1. Let Π be a closed hyperplane in a Banach space Z, $z_0 \in \Pi$, $\alpha > 0$ and let $\mathcal{C}'_{\Pi,\alpha}(z_0)$ be the corresponding circular co-cone. Then

$$\mathcal{C}'_{\Pi,\alpha}(z_0) = \{z ; |\langle z - z_0, z^* \rangle| \leq \frac{\alpha \cdot \langle u, z^* \rangle}{d(u, \Pi)} \cdot \|z - z_0\|\}$$

for every $u \in Z \setminus \Pi$ and $z^* \in Z^*$ such that $\langle z - z_0, z^* \rangle = 0$

whenever $x \in \Pi$.

Proof. Let $x' \in \mathcal{C}'_{\Pi,\alpha}(x_0)$; then $d\left(\frac{x'-x_0}{\|x'-x_0\|}, \Pi\right) \leq \alpha$.

Let u and x^* be as in the lemma and denote $\Pi_\tau = \{x: \langle x-x_0, x^*\rangle = \tau\}$. The set Π_τ is a hyperplane and it can be easily shown that $\Pi_\tau = \Pi + \frac{\tau}{\langle u, x^*\rangle} \cdot u$ and $d(\Pi_\tau, \Pi) = \frac{d(u,\Pi)}{|\langle u, x^*\rangle|} \cdot |\tau|$. Hence $\frac{x'-x_0}{\|x'-x_0\|} \in \Pi_{\tau'}$, where $|\tau'| \leq \frac{|\langle u, x^*\rangle|}{d(u,\Pi)} \cdot \alpha$, whence the result. The converse can be proved similarly.

Now, let X, Y be Banach spaces and denote by $\mathbb{G}$ the system of graphs of all continuous linear mappings from X into Y . Hence, every $\Pi \in \mathbb{G}$ is a closed linear subspace of $X \times Y$.

Definition. Let X, Y be Banach spaces, $A \subset X$, $F: A \rightarrow Y$, x_0 an interior point of A and let Π be a flat in $X \times Y$. The flat Π is said to be tangent to the graph $\mathcal{G}(F)$ of F at the point $(x_0, F(x_0))$ iff the following two conditions are fulfilled:

(i) $\Pi - (x_0, F(x_0)) \in \mathbb{G}$

(ii) For each $\alpha > 0$ there is $\varkappa(\alpha) > 0$ such that

$$\mathcal{G}(F) \cap B_{\varkappa(\alpha)}(x_0, F(x_0)) \subset \bigcap_{H \in \mathbb{H}} \mathcal{C}'_{H,\alpha}(x_0, F(x_0))$$

where $B_\varkappa(x_0, F(x_0)) = \{z \in X \times Y: \|z-(x_0, F(x_0))\| < \varkappa\}$ and $\mathbb{H}$ is the system of all closed hyperplanes H in $X \times Y$ having the property $\Pi \subset H$.

Lemma 2. If Π is a closed flat in a Banach space Z and $\mathbb{H}$ is the system of all closed hyperplanes H in

Z such that $\Pi \subset H$, then $\bigcap_{H \in \mathbb{H}} H = \Pi$.

<u>Proof</u>. Assume that there is $z' \in \bigcap_{H \in \mathbb{H}} H$ such that $z \notin \Pi$ and let z_0 be an arbitrary point of Π . By Hahn-Banach Theorem, there is $z^* \in Z^*$ such that $\| z^* \| = 1$, $\langle z - z_0, z^* \rangle = 0$ whenever $z \in \Pi$ and $\langle z' - z_0, z^* \rangle = d(z', \Pi) > 0$. Denote $H_{z^*} = \{ z \in Z : \langle z - z_0 , z^* \rangle = 0 \}$; H_{z^*} is a closed hyperplane and $\Pi \subset H_{z^*}$, hence $H_{z^*} \in \mathbb{H}$. It implies that $z' \in H_{z^*}$ but it is contradictory to $\langle z' - z_0 , z^* \rangle > 0$. The converse inclusion is trivial.

Let us remark that the notion of a tangent flat to a graph defined above agrees in finitely dimensional spaces with the analogical Roetman's notion and moreover, the condition (i) implies the tangent flat Π is not parallel to the space Y (it means the flats Π and $\{0_X\} \times Y$ are not parallel; two flats Π_1 and Π_2 are said to be parallel iff $(\Pi_1 - z_1) \subset (\Pi_2 - z_2)$ or $(\Pi_2 - z_2) \subset (\Pi_1 - z_1)$ for some $z_1 \in \Pi_1$ and $z_2 \in \Pi_2$). Using this more general notion of a tangent, we can now prove the following theorem that is formally identical with the Roetman's theorem quoted above but that characterizes F-differentiability of mappings also in infinitely dimensional spaces (we write F-differentiability for Fréchet differentiability etc.).

<u>Theorem 1</u>. Let X, Y be Banach spaces, $A \subset X$, $F: A \longrightarrow Y$ and let x_0 be an interior point of A . The mapping F possesses the F-derivative at the point x_0 if and only if there exists a tangent flat to the graph of F at the point $(x_0 , F(x_0))$.

Proof. We shall consider the sum norm in $X \times Y$ (that is the norm defined by $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$) but it is not essential - an arbitrary equivalent norm can be considered. Denote $Z = X \times Y$, $y_0 = F(x_0)$ and $z_0 = (x_0, y_0)$.

1) Suppose F possesses the F-derivative $F'(x_0)$ at x_0 and set

$$\Pi = \{(x,y) \in Z : y = y_0 + F'(x_0)(x - x_0)\} .$$

Evidently, $(\Pi - (x_0, y_0)) = \mathcal{G}(F'(x_0)) \in \mathbb{G}$. Set

$$P = \{z^* \in Z^* : \|z^*\| = 1, \langle z - z_0, z^* \rangle = 0 \text{ whenever } z \in \Pi\}$$

and denote $\mathbb{H} = \{H_{z^*} : z^* \in P\}$ where $H_{z^*} = \{z \in Z : \langle z - z_0, z^*\rangle = 0\}$.
It is $\Pi \subset H$ for every $H \in \mathbb{H}$ and, conversely, every hyperplane H in Z such that $\Pi \subset H$, belongs to $\mathbb{H}$. Indeed, there is $z^*_H \in Z$ for every $H \supset \Pi$ such that $\|z^*_H\| = 1$ and $\langle z - z_0, z^*_H \rangle = 0$ whenever $z \in H$; since $\Pi \subset H$, it is $z^*_H \in P$ and hence $H \in \mathbb{H}$. Moreover, it is $\bigcap_{z^* \in P} H_{z^*} = \Pi$ by Lemma 2.

To prove Π is a tangent flat to $\mathcal{G}(F)$ at x_0 , it remains to verify the condition (ii). Suppose to the contrary that there is $\alpha > 0$ and $z_m \in \mathcal{G}(F)$ such that $\|z_m - z_0\| \le \frac{1}{m}$ and $z_m \notin \bigcap_{H \in \mathbb{H}} \mathcal{C}'_{H,\alpha}(z_0)$ for $m = 1, 2, \dots$.
It means there is $H_m \in \mathbb{H}$ for every m such that

$$z_m \notin \mathcal{C}'_{H_m,\alpha}(z_0) .$$

Choosing u_m and z^*_m in the manner described in the construction before Lemma 1, we can see that $H_{z^*} = H_m$ and

$$|\langle z_m - z_0, z_m^* \rangle| > \alpha \|z_m - z_0\|$$

for every m by (1). Since $\Pi \subset H_{z_m^*}$ for every m, it follows

$$|\langle z_m - z, z_m^* \rangle| \geq |\langle z_m - z_0, z_m^* \rangle| - |\langle z_0 - z, z_m^* \rangle| > \alpha \|z_m - z_0\|$$

for all $z \in \Pi$ and hence

(2) $(\|y_m - y\| + \|x_m - x\|) > \alpha(\|y_m - y_0\| + \|x_n - x_0\|) \geq \alpha \|x_n - x_0\|$

where $z_m = (x_m, y_m)$ and $z = (x, y) \in \Pi$.

Now, set $z'_m = (x'_m, y'_m)$ where $x'_m = x_m$, $y'_m = y_0 + F'(x_0)(x_m - x_0)$. Evidently $z'_m \in \Pi$ and so it follows from (2) that

$$\|y_m - y_0 - F'(x_0)(x_m - x_0)\| > \alpha \|x_m - x_0\|$$

for all m. However, it contradicts our assumption on F-differentiability of F at x_0 because $\|x_m - x_0\| \leq \|z_m - z_0\| \leq \frac{1}{m}$.

2) On the other hand, suppose now that there is a tangent flat Π to $\mathcal{G}(F)$ at $z_0 = (x_0, F(x_0))$ and prove that F is F-differentiable at x_0.

According to (i), there is a continuous linear mapping $L : X \longrightarrow Y$ such that $\Pi = \{(x, y) \in Z : y = y_0 + L(x - x_0)\}$. Define the sets P, H_{z^*} and the system $\mathbb{H}$ in the same manner as in the first part of our proof. Then $\mathbb{H}$ is the system of all hyperplanes in Z containing Π, again, and it is $\bigcap_{H \in \mathbb{H}} H = \Pi$ by Lemma 2.

Now, let $\alpha > 0$ be an arbitrary number and let $z' \in \bigcap_{H \in \mathbb{H}} \mathcal{C}'_{H,\alpha}(z_0)$. Then by our Lemma 1,

(3) $$|\langle z' - z_0, z^* \rangle| \leq \frac{\alpha \langle u_{z^*}, z^* \rangle}{d(u_{z^*}, H_{z^*})} \cdot \|z' - z_0\|$$

for all $x^* \in P$ and all $u_{x^*} \in Z \setminus H_{x^*}$. According to the Hahn-Banach Theorem, there is $x'^* \in Z^*$ such that $\|x'^*\| = 1$, $\langle x - x_0, x'^* \rangle = 0$ whenever $x \in \Pi$ and

(4) $$\langle x' - x_0, x'^* \rangle = d(x', \Pi) .$$

It is $x'^* \in P$ and hence, choosing $u_{x'^*}$ in (3) so that $\|u_{x'^*}\| = 1$ and $d(u_{x'^*}, H_{x'^*}) \geq \frac{1}{2}$ (such $u_{x'^*}$ exists by the well-known theorem of F. Riesz, see e.g. [13]), we obtain from (3)

(5) $$|\langle x' - x_0, x'^* \rangle| \leq 2\alpha \|x' - x_0\| .$$

In view of the definition of a distance as an infimum, we can find $x'' \in \Pi$ so that

$$|d(x', \Pi) - \|x' - x''\|| < \alpha \|x' - x_0\| ,$$

whence

(6) $$\langle x' - x_0, x'^* \rangle \geq \|x' - x''\| - \alpha \|x' - x_0\|$$

by (4). Denoting $x' = (x', y')$ and $x'' = (x'', y'')$ we have $y'' = y_0 + L(x'' - x_0)$ and so it follows from (5) and (6) that

(7) $$\|y' - y_0 - L(x' - x_0)\| \leq \|y' - y_0 - L(x'' - x_0)\| + \|L\| \cdot \|x'' - x'\| \leq$$
$$\leq (1 + \|L\|) \|x' - x''\| \leq (1 + \|L\|)(\langle x' - x_0, x'^* \rangle + \alpha \|x' - x_0\|) \leq$$
$$\leq 3\alpha (1 + \|L\|) \|x' - x_0\| .$$

This inequality implies that

$$\|y' - y_0\| - \|L(x' - x_0)\| \leq 3\alpha (1 + \|L\|)(\|x' - x_0\| + \|y' - y_0\|)$$

whence

$$(8) \qquad \|y'-y_0\| \leq \frac{\|L\|+3\alpha(1+\|L\|)}{1-3\alpha(1+\|L\|)} \cdot \|x'-x_0\|$$

assuming $\alpha < \dfrac{1}{3(1+\|L\|)}$. It follows now from this relation and (7) that

$$(9) \qquad \|y'-y_0-L(x'-x_0)\| \leq \frac{3\alpha(1+\|L\|)^2}{1-3\alpha(1+\|L\|)} \cdot \|x'-x_0\|$$

for every $z'=(x',y') \in \bigcap_{H\in\mathbb{H}} \mathscr{C}'_{H,\alpha}(z_0)$ if $\alpha < \dfrac{1}{3(1+\|L\|)}$.

Now, let $\varepsilon > 0$ be an arbitrary given number; we can assume that $\varepsilon < 1$. Set $\alpha = \dfrac{\varepsilon}{3(1+\|L\|)(1+\|L\|+\varepsilon)}$ and let $r(\alpha)$ be a number corresponding to this α according to (ii); note that $\alpha < \dfrac{1}{3(1+\|L\|)}$. Choose

$\delta \leq \dfrac{1-3\alpha(1+\|L\|)}{1+\|L\|+3\alpha(1+\|L\|)} \cdot \dfrac{r(\alpha)}{2}$ so small (but positive) to be $\{x \in X : \|x-x_0\| \leq \delta\} \subset A$; it is $0<\delta<\dfrac{r(\alpha)}{2}$ and hence $\|x-x_0\| < \dfrac{r(\alpha)}{2}$ whenever $\|x-x_0\|<\delta$. If $(x,y) \in \bigcap_{H\in\mathbb{H}} \mathscr{C}'_{H,\alpha}(z_0)$ then $\|x-x_0\|<\delta$ implies

$$\|y-y_0\| \leq \frac{\|L\|+3\alpha(1+\|L\|)}{1+\|L\|+3\alpha(1+\|L\|)} \cdot \frac{r(\alpha)}{2} < \frac{r(\alpha)}{2}$$

by (8) and so $\Delta \cap \bigcap_{H\in\mathbb{H}} \mathscr{C}'_{H,\alpha}(z_0) \subset B_{r(\alpha)}(z_0)$ where $\Delta = \{(x,y) \in X\times Y : \|x-x_0\| \leq \delta\}$. Therefore,

$$\mathcal{G}(F) \cap \Delta \cap \bigcap_{H\in\mathbb{H}} \mathscr{C}'_{H,\alpha}(z_0) \subset \mathcal{G}(F) \cap B_{r(\alpha)}(z_0) \subset \bigcap_{H\in\mathbb{H}} \mathscr{C}'_{H,\alpha}(z_0)$$

by (ii) and hence

(10) $$\mathcal{G}(F) \cap \Delta \subset \bigcap_{H \in \mathbb{H}} \mathcal{C}'_{H,\alpha}(x_0) \ .$$

It follows from (9) and (10) that

$$\|F(x) - F(x_0) - L(x - x_0)\| \leq \varepsilon \|x - x_0\|$$

for all $x \in A$, $\|x - x_0\| \leq \delta$, which implies that F possesses the F-derivative $F'(x_0) = L$ at the point x_0. This completes the proof.

(1.3) Let $\mathfrak{S}$ be a system of sets $C \subset X$ that are star-shaped with respect to 0 and such that there is $C \in \mathfrak{S}$ with $diam\, C < r$ for every $r > 0$.

<u>Definition</u>. Let X, Y be Banach spaces, $A \subset X$, $F: A \to Y$, $x_0 \in Int\, A$ (interior of A) and let Π be a flat in $X \times Y$. The flat Π is said to be $\mathfrak{S}$-tangent to the graph $\mathcal{G}(F)$ of F at x_0 iff the two following conditions are fulfilled:

(i') $\Pi - (x_0, F(x_0)) \in \mathbb{G}$ where $\mathbb{G}$ is as in (1.2)

(ii') There are $r(\alpha) > 0$ and $C_\alpha \in \mathfrak{S}$ for each $\alpha > 0$ such that

(11) $$\mathcal{G}(F) \cap [(x_0, F(x_0)) + C_\alpha \times B^Y_{r(\alpha)}] \subset \bigcap_{H \in \mathbb{H}} \mathcal{C}'_{H,\alpha}(x_0, F(x_0))$$

where $B^Y_r = \{y \in Y : \|y\| < r\}$ and $\mathbb{H}$ is the system of all closed hyperplanes H in $X \times Y$ such that $\Pi \subset H$.

Particularly, we denote by $\mathfrak{S}_0$ the system of all subsets of X that are star-shaped with respect to 0 and by $\mathfrak{S}_1$ the system of all $C \in \mathfrak{S}_0$ such that $0 \in Int\, C$. Our

Theorem 1 can be now rewritten as follows:

<u>Theorem 1'</u>. A mapping $F: A \to Y$ $(A \subset X)$ possesses a Fréchet derivative at $x_0 \in \mathrm{Int}\, A$ if and only if $\mathcal{G}(F)$ possesses a $\mathcal{G}_1$ -tangent flat at x_0 .

The following theorem can be proved in a similar way.

<u>Theorem 2</u>. A mapping $F: A \to Y$ $(A \subset X)$ possesses a Gâteaux derivative at $x_0 \in \mathrm{Int}\, A$ if and only if $\mathcal{G}(F)$ possesses a $\mathcal{G}_0$ -tangent flat at x_0 .

Note that it is possible to characterize also the differentiability of a mapping $F: A \to Y$ $(A \subset X)$ at $x_0 \in A$ relative to a set $M \subset A$; to this aim, only the change of C_α in (11) for $C_\alpha \cap M$ is needed.

2. <u>Characterization in terms of tangent cones</u>

(2.1) Another approach to the geometric characterization of differentiability was studied by T.M. Flett, who introduced in [3] and [4] the notions of tangent rays and cones. We recall his definitions:

Let X be a Banach space, $A_0 \subset X$, x_0 be a cluster point of A_0 and denote $A = A_0 \setminus \{x_0\}$. If the limit

$$\lim_{\substack{x \to x_0 \\ x \in A}} \frac{x - x_0}{\|x - x_0\|} = u \in X$$

exists then the ray in X with the beginning at x_0 and the direction u (i.e. the set $\{x \in X: x = x_0 + \lambda u,\ \lambda \geq 0\}$) is called the tangent ray to A_0 at x_0 .

Let $S \subset X$, $x_0 \in \overline{S}$. The union of all tangent rays at x_0

to all subsets $A_0 \subset S$ for which such a ray exists is said to be the tangent cone to S at x_0 ; if there is no such $A_0 \subset S$ then we define the tangent cone to S at x_0 to be the one-point set $\{x_0\}$.

Flett proved in his paper [4] the following theorems (see [4], Theorem 1(i) and Theorem 5):

<u>Theorem A.</u> Let X, Y be Banach spaces, $D \subset X$, $x_0 \in$ $\in \operatorname{Int} D$, let $F: D \to Y$ be a mapping F - differentiable at x_0 and denote $\varphi(x) = F(x_0) + F'(x_0)(x - x_0)$ for $x \in X$. Then the tangent cone to $\mathcal{G}(F)$ at the point $(x_0, F(x_0))$ equals to $\mathcal{G}(\varphi)$.

<u>Theorem B.</u> Let X, Y be finitely dimensional spaces, $D \subset X$, $x_0 \in \operatorname{Int} D$, let $F: D \to Y$ be a mapping continuous at x_0 and suppose the tangent cone to $\mathcal{G}(F)$ at $(x_0, F(x_0))$ is contained in a set $(x_0, F(x_0)) + \mathcal{G}(L)$ where $L: X \to Y$ is a continuous linear mapping. Then the mapping F has the Fréchet derivative $F'(x_0)$ at x_0 and $F'(x_0) = L$.

Flett [4] put the question (see also [5]) whether it would be possible to define F-differentiability by means of some tangent cone also in infinitely dimensional spaces. We show in the next paragraph that using tangent cones in the sense of Flett [4], such total characterization of F-differentiability cannot be given, even under very strong restrictions.

(2.2) Consider the following example. Let X be a real line, Y a real infinitely dimensional Hilbert space, $\{e_n\}$

an infinite orthonormal sequence in Y and define a mapping $F: X \to Y$ as follows:

$$\begin{cases} F(x) = 0 \quad \text{for} \quad |x| = \dfrac{1}{2n-1}, \quad n = 1, 2, \dots \\ F(x) = \dfrac{1}{n} e_n \quad \text{for} \quad |x| = \dfrac{1}{2n}, \quad n = 1, 2, \dots \\ F(x) = 0 \quad \text{for} \quad |x| \geq 1 \\ F(x) \quad \text{is linear in each of the intervals} \\ \quad [\dfrac{1}{2n+1}, \dfrac{1}{2n}], \ [\dfrac{1}{2n}, \dfrac{1}{2n-1}], [\dfrac{-1}{2n-1}, \dfrac{-1}{2n}] \\ \quad \text{and } [\dfrac{-1}{2n}, \dfrac{-1}{2n+1}], \quad n = 1, 2, \dots \end{cases}$$

The mapping F is locally Lipschitzian and maps the whole 1-dimensional space X (the real line) into the Hilbert space Y. It is $F(0) = 0$ and we show that the tangent cone to $\mathcal{G}(F)$ at the point $(0, 0) \in X \times Y$ is the line $L_X = \{(x, y) \in X \times Y : y = 0\}$.

Indeed, let A be a subset of $\mathcal{G}(F)$ such that $z_0 = (0, 0) \in \bar{A} \setminus A$ and let $z_n = (x_n, F(x_n))$ be a sequence in A which converges to z_0 (we shall consider the sum norm in $X \times Y$ as in the preceding paragraphs). We can suppose without loss of generality that $x_n > 0$ for all $n = 1, 2, \dots$ and that there is at most one x_n in every interval $[\frac{1}{2k+1}, \frac{1}{2k-1}]$ $(k = 1, 2, \dots)$; denote by $i(n)$ such a number that $x_n \in [\frac{1}{2i(n)+1}, \frac{1}{2i(n)-1}]$ for $n = 1, 2, \dots$.

Now, every x_n can be expressed in one of the forms

(12a) $$x_n = \frac{1}{2i(n)+1} + \vartheta_n \cdot \left(\frac{1}{2i(n)} - \frac{1}{2i(n)+1}\right), \ \vartheta_n \in [0,1]$$

or

(12b) $$x_n = \frac{1}{2i(n)} + \vartheta_n \cdot \left(\frac{1}{2i(n)-1} - \frac{1}{2i(n)}\right), \ \vartheta_n \in [0,1] .$$

Assume for instance that all x_n are expressed in the form (12a) (other cases would be processed similarly) and that $x_n < x_m$ whenever $n < m$; then

$$F(x_n) = \frac{\vartheta_n}{i(n)} \cdot e_{i(n)}$$

and

$$\|z_n\| = \|x_n\| + \|F(x_n)\| = \frac{2i(n) + 4\vartheta_n i(n) + 3\vartheta_n}{2i(n) \cdot (2i(n)+1)} .$$

Denote P the projection of $(X \times Y) \setminus \{z_0\}$ onto $S = \{z \in X \times Y : \|z\| = 1\}$, i.e. $P(z) = \frac{z}{\|z\|}$ for every $z \in X \times Y, z \neq (0,0)$. It is easy to calculate that for every $n, m, m > n$,

(13) $$\|P(z_n) - P(z_m)\| = \left\| \frac{x_n}{\|z_n\|} - \frac{x_m}{\|z_m\|} \right\| + \left\| \frac{F(x_n)}{\|z_n\|} - \frac{F(x_m)}{\|z_m\|} \right\| =$$

$$= \left[\frac{2i(n) + \vartheta_n}{2i(n) + 3\vartheta_n + 4i(n)\vartheta_n} - \frac{2i(m) + \vartheta_m}{2i(m) + 3\vartheta_m + 4i(m)\vartheta_m} \right] +$$

$$+ \left[\left(2\vartheta_n \cdot \frac{2i(n)+1}{2i(n)+3\vartheta_n+4i(n)\vartheta_n}\right)^2 + \left(2\vartheta_m \cdot \frac{2i(m)+1}{2i(m)+3\vartheta_m+4i(m)\vartheta_m}\right)^2 \right]^{\frac{1}{2}} .$$

If we denote the first term on the right side of (13) by T_1 then the following estimate holds:

$$0 \le T_1 = \left[\frac{1}{1+2\vartheta_n+\frac{3\vartheta_n}{2i(n)}} - \frac{1}{1+2\vartheta_m+\frac{3\vartheta_m}{2i(m)}}\right] +$$

$$+\left[\frac{\vartheta_n}{2i(n)+(3+4i(n))\vartheta_n}\right] - \frac{\vartheta_m}{(2+4\vartheta_m)i(m)+3\vartheta_m}\Big] \le$$

$$\le \left|2\vartheta_m+\frac{3\vartheta_m}{2i(m)}-2\vartheta_n-\frac{3\vartheta_n}{2i(n)}\right| + \left|\frac{\vartheta_n}{2i(n)}-\frac{\vartheta_m}{6i(m)+3}\right| \le$$

$$\le \left(4+\frac{44+\frac{21}{i(n)}}{12i(n)+6}\right)\cdot \vartheta_{max}$$

where $\vartheta_{max} = max(\vartheta_n, \vartheta_m)$. The second term on the right side of (13) - denote it by T_2 - can be estimated as follows:

$$T_2 \le 2\vartheta_{max}\cdot\left[\left(\frac{2i(n)+1}{2i(n)}\right)^2 + \left(\frac{2i(m)+1}{2i(m)}\right)^2\right]^{\frac{1}{2}} \le$$

$$\le \left(2+\frac{1}{i(n)}\right)\sqrt{2}\cdot\vartheta_{max}$$

$$T_2 \ge 2\vartheta_{min}\cdot\left[\left(\frac{2i(n)+1}{6i(n)+3}\right)^2+\left(\frac{2i(m)+1}{6i(m)+3}\right)^2\right]^{\frac{1}{2}} \ge \frac{2}{3}\sqrt{2}\cdot\vartheta_{min}$$

where $\vartheta_{min} = min(\vartheta_n, \vartheta_m)$ and ϑ_{max} is as above. We conclude from these estimates and (13) that

$$(14) \quad c_0\cdot min(\vartheta_n,\vartheta_m) \le \|P(z_n)-P(z_m)\| \le c(n)\cdot max(\vartheta_n,\vartheta_m)$$

for every $n, m, n < m$ where $c_0 > 0$, $c(n) > 0$ for all n and $\lim_{n\to\infty} c(n) = c_1 > 0$. We can see from (14) that the sequence $\{P(z_n)\}$ converges if and only if the sequence $\{z_n\}$ has the property $\vartheta_n \longrightarrow 0$. If this is the case, then denoting $z^* = (1,0) \in S \subset X \times Y$, it holds

$$\|P(z_n)-z^*\| = \left\|\frac{x_n}{\|z_n\|}-1\right\| + \left\|\frac{F(x_n)}{\|z_n\|}\right\| =$$

$$= \left[\frac{1}{1+2\vartheta_m+\frac{3\vartheta_m}{2i(m)}} - \frac{\vartheta_m}{2i(m)+4i(m)\vartheta_m+3\vartheta_m} - 1\right] + 2\vartheta_m \cdot \frac{1+\frac{1}{2i(m)}}{1+2\vartheta_m+\frac{3\vartheta_m}{2i(m)}}$$

and so $P(z_m) \to z^*$ if $m \to \infty$. Thus we have proved that z^* is the only limit point of $P(A - z_0)$ for arbitrary $A \subset \mathcal{G}(F)$ with $z_0 = (0, 0) \in \bar{A} \setminus A$. Hence according to the Flett's definition quoted above, the line L_X is the tangent cone to $\mathcal{G}(F)$ at z_0.

On the other hand, it is evident that the Fréchet derivative of F at $0 \in X$ does not exist. In fact, if there is the derivative of F at 0 it would be equal to zero-operator N by Theorem A and by the assertion just been proved. However, choosing $x_m = \frac{1}{2m}$ $(m = 1, 2, \dots)$ we have $x_m \to x_0 = 0$ and

$$\frac{\|F(x_m)-F(x_0)-N(x_m-x_0)\|}{\|x_m-x_0\|} = \frac{\|F(x_m)\|}{\|x_m\|} = \frac{\frac{1}{m}}{\frac{1}{2m}} = 2$$

for all m, which contradicts the definition of the F-derivative.

The reason why the Flett's notion of a tangent cone is not adequate to the characterization of F-differentiability, is the following: In the Flett's definition of a tangent cone, only such sequences $\{z_m\} \subset \mathcal{G}(F)$, $z_m \to z_0$ are taken into account for which the sequences $\{P(z_m)\}$ are convergent while on the other hand, all sequences $\{z_m\} \subset \mathcal{G}(F)$, $z_m \to z_0$ are considered in the definition of an F-derivative. This difference is not essential in the case of finitely dimensional spaces because the set $\{P(z_m)\}$

is then compact for every $\{x_n\}$ and hence every sequence $\{P(x_n)\}$ has a convergent subsequence. In infinitely dimensional spaces, this difference is unfortunately essential and in order to make the total characterization of differentiability possible, we must modify the Flett's definition in an appropriate manner. In this respect, see [11] for concrete results.

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