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QUASICOMPLEMENTED LATTICES

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Abstract: Let L be a 0-distributive lattice. Then L is quasicomplemented if and only if each minimal prime ideal in the lattice $\mathcal{J}(L)$ of ideals in L contrasts to a minimal prime ideal in L. A necessary and sufficient condition is also given for the contraction map to be a bijection of the set of minimal prime ideals of $\mathcal{J}(L)$ onto the set of minimal prime ideals of L. Amongst distributive lattices, a new characterization of quasicomplemented lattices is presented in terms of "lifting" dense elements modulo the smallest congruence having a minimal prime ideal as its kernel.

Key words: 0-distributive, quasicomplemented, minimal prime ideal, lattice of ideals, compact space, extremally disconnected space.

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1. 0-distributivity. According to Varlet [9], a lattice L with least element 0 is called 0-distributive if it satisfies the condition: $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$, for any a, b, c in L. This concept is both a generalization of pseudocomplementation and distributivity. It is equivalent to the condition that $\mathcal{J}^* = \{x \in L : x \wedge j = 0$ for each $j \in \mathcal{J}\}$ is a lattice-ideal for each ideal or non-empty subset $\mathcal{J}$ of L and hence, as was noted by Varlet [9, Theorem 1], to the condition that the lattice $\mathcal{J}(L)$ of ideals in L is pseudocomplemented.

By a minimal prime ideal of a lattice or semigroup with 0

we mean a prime ideal (necessarily a proper subset) which is minimal amongst the prime ideals ordered by set-inclusion. For further details on minimal prime ideals we refer to [5] and [4]. The following theorem shows that there are sufficiently many minimal prime ideals in a 0-distributive lattice. It is a consequence of Keimel's general theory of minimal prime ideals, see [4, Theorem C. Corollary]. Most of it is given in [2, Proposition 7.26, p.92]. However, we give an alternative proof based on Kist's work [5], describing prime ideals in a commutative semigroup.

1.1. <u>Proposition</u>. For a lattice L with 0, the following conditions are equivalent:

(a) L is 0-distributive.

(b) The minimal prime ideals of the semigroup $(L;\wedge,0)$ are minimal prime ideals of the lattice L.

(c) For each $a \in L$ with $a \neq 0$, there is a minimal prime ideal P such that $a \notin P$.

(d) The zero ideal of the lattice L is an intersection of prime ideals.

<u>Proof</u>. (a) $\Longrightarrow$ (b). By [5, Corollary 1.4 and Lemma 3.1] the semigroup $(L;\wedge,0)$ possesses minimal prime ideals and a prime ideal P is a minimal prime ideal if and only if, for each $a \in P$, there exists $b \notin P$ such that $a \wedge b = 0$. Thus, if P is a minimal prime in $(L;\wedge,0)$ and $a_1, a_2 \in P$ then $a_1 \wedge b_1 = 0 = a_2 \wedge b_2$ for some b_1, $b_2 \notin P$. As P is prime $b_1 \wedge b_2 \notin P$ and yet $(a_1 \vee a_2) \wedge (b_1 \wedge b_2) = 0 \in P$, by 0-distributivity. It

follows that $a_1 \vee a_2 \in P$ and so P is a lattice ideal.

(b) $\Longrightarrow$ (c) holds in the lattice $(L; \vee, \wedge, 0)$ since (b) $\Longrightarrow$ (c) holds in the semigroup $(L; \wedge, 0)$ by [5, Lemma 1.2].

(c) $\Longrightarrow$ (d) is trivial, while (d) $\Longrightarrow$ (a) holds since $a \wedge b = 0 = a \wedge c$ and $\{0\} = \cap P_i$, for suitable prime ideals P_i, imply $a \wedge (b \vee c) = 0$.
Otherwise, $a \wedge (b \vee c) \notin P_j$ for some j and so $a \notin P_j$, whence $b, c \in P_j$ as $a \wedge b = 0 = a \wedge c$ and P_j is prime. But then $b \vee c \in P_j$ yields an impossibility.

Since any prime ideal of the lattice $(L; \vee, \wedge, 0)$ is a prime ideal of the semigroup $(L; \wedge, 0)$, Theorem 1.1 shows that a lattice L with 0 is 0-distributive if and only if the minimal prime ideals of $(L; \vee, \wedge, 0)$ are precisely the minimal prime ideals of $(L; \wedge, 0)$.

Following Varlet [9], a lattice L with 0 is called <u>quasicomplemented</u> if, for each $x \in L$, there is an element $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense. Of course, an element $d \in L$ is <u>dense</u> if $\{a \in L : a \wedge d = 0\} = \{0\}$. In general the element x' is highly non-unique. Besides being 0-distributive, a pseudocomplemented lattice L is quasicomplemented - we may simply choose x' to be x^*, the pseudocomplement of x.

For an element x in a lattice L with 0, let $(x] = \{a \in L : a \leq x\}$ denote the principal ideal generated by x. Then, as was established by Varlet [9, Theorem 10], a 0-distributive lattice L is quasicomplemented if and only if, for each $x \in L$, there exists $x' \in L$ such that $(x]^{**} = (x']^{*}$.

Let $Min(L)$ denote the set of all minimal prime ideals of a 0-distributive lattice L. We may turn $Min(L)$ into a Hausdorff topological space by endowing it with the so-called hull-kernel topology which has the sets of the form $\{P \in Min(L) : x \notin P\}$ $(x \in L)$ as a base for the open sets. For details on this topology see [5], [4], [6] and [8]. Applying Theorem 1.1 and the main theorem of [8], we immediately obtain

1.2. <u>Proposition</u>. A 0-distributive lattice L is quasicomplemented if and only if $Min(L)$ is a compact Hausdorff space.

Of course, 1.2 is also a consequence of [4, Proposition 5.10, Corollary]. Proposition 1.2, together with the next result, constitute our tools for proving the main results of this paper.

1.3. <u>Proposition</u>. Let L be a quasicomplemented 0-distributive lattice. Then $Min(L)$ is extremally disconnected if and only if for each ideal J in L, there exists $y \in L$ such that $J^{*} = (y]^{*}$.

Recall that a topological space is extremally disconnected if and only if the closure of each open set is open. position 1.3 can be obtained by adapting [1, Theorem 4.4] rom ring-notation to lattice-notation. There are no hidden difficulties . Alternatively, it is easily proved that, for a quasicomplemented 0-distributive lattice L, the space of minimal prime ideals is the Stone representation space for the Boolean algebra of all ideals of the form $(x]^{**}$ $(x \in L)$. That we have a Boolean algebra can be seen from either [9, Main Theorem, p.156] or [7, Theorem 1]. The assertion then follows from the well-known fact that a Boolean algebra is complete if and only if its representation space is extremally disconnected and the observation that the Boolean algebra of ideals $(x]^{**}$ is complete if and only if the condition of 1.3 obtains. This last observation follows from [7, Theorem 2, Corollary].

1.4. <u>Lemma</u>. For any 0-distributive lattice L, $Min(\mathcal{J}(L))$ is a compact Hausdorff extremally disconnected space.

<u>Proof</u>. Since L is 0-distributive, $\mathcal{J}(L)$ is pseudocomplemented and so $Min(\mathcal{J}(L))$ is compact and Hausdorff because of 1.2. For a non-empty subset of $\mathcal{J}$ of $\mathcal{J}(L)$, $\{J \in \mathcal{J}(L) : J \cap K = (0]$ for each $K \in \mathcal{J}\} =$ $= \{J \in \mathcal{J}(L) : J \cap Y = 0$, where $Y = \bigvee\{K : K \in \mathcal{J}\}\}$ and so the rest follows from 1.3.

2. <u>Main Theorems</u>. For a 0-distributive lattice L and a prime ideal P in $\mathcal{J}(L)$, $c(P)$ denotes the set-theoretical union of all ideals (of L) which are in P, while for a prime ideal Q in L, $\mathfrak{p}(Q)$ denotes the set

$\{J \in \mathcal{J}(L) : J \subseteq Q\}$. If we identify the members of L with the corresponding principal ideals which they generate and thereby identify L with a sublattice of $\mathcal{J}(L)$ then $c(P) = L \cap P$ for each prime ideal P in $\mathcal{J}(L)$. That is, $c(P)$ is then nothing more than the <u>contraction</u> of P to the sublattice L of $\mathcal{J}(L)$. Thus, in the statements of the main theorems we shall speak of <u>contractions</u> of minimal prime ideals in $\mathcal{J}(L)$ to L though, for the sake of clarity, it will be convenient to use our initial description of $c(P)$ $(P \in \mathcal{M}in(\mathcal{J}(L)))$ in the proofs.

For a prime ideal P in $\mathcal{J}(L)$ and a prime ideal Q in L, it is easy to see that $c(P)$ and $p(Q)$ are prime ideals in L and $\mathcal{J}(L)$, respectively. This was observed by Katriňák [3] in the case of distributive lattices. In fact the main theorems were inspired by [3, Lemma 12 and Theorem 5]. They not only explain [3, Lemma 12] but also clarify Theorem 5 of [3], wherein Katriňák gives a necessary and sufficient condition, involving contractions of minimal prime ideals, for the lattice of ideals of a distributive lattice with 0 and 1 to be a Stone lattice.

2.1. <u>Theorem</u>. A 0-distributive lattice L is quasicomplemented if and only if each minimal prime ideal in $\mathcal{J}(L)$ contracts to a minimal prime ideal in L.

<u>Proof</u>. Suppose L is quasicomplemented. Let $P \in \mathcal{M}in(\mathcal{J}(L))$ and $x \in c(P)$. Then, $(x] \in P$. Choose $x' \in L$ such that $x \vee x'$ is dense and $x \wedge x' = 0$. We claim that $x' \notin c(P)$. Otherwise, $x' \in c(P)$, $(x'] \in P$, and $(x \vee x'] = (x] \vee (x'] \in P$, and so the dense ele-

ment $(x \vee x']$ of $\mathcal{J}(L)$ is in the minimal prime ideal P. This contradicts the following characterization of a minimal prime ideal in a 0-distributive lattice L: a prime ideal Q in a 0-distributive lattice is a minimal prime ideal if and only if, for each $a \in Q$, there exists $b \in L \setminus Q$ such that $a \wedge b = 0$. This characterization which will also be freely used below, follows from 1.1 and the proof of (a) $\Longrightarrow$ (b) in 1.1. Thus, it is indeed the case that $x' \notin c(P)$. Since $c(P)$ is a prime ideal it follows that it is a minimal prime ideal.

Conversely, suppose $c(P) \in Min(L)$ for each $P \in Min(\mathcal{J}(L))$. Then, we have a function $c: Min(\mathcal{J}(L)) \longrightarrow Min(L)$ such that $c: P \longmapsto c(P)$ for each $P \in Min(\mathcal{J}(L))$. This function is a surjection. For if $Q \in Min(L)$, $p(Q)$ is a prime ideal in $\mathcal{J}(L)$ and so, by Zorn's lemma, it contains at least one minimal prime ideal P. Then $c(P) = Q$. Since, if $a \in c(P)$ then $(a] \in P \subseteq p(Q)$ and so $(a] \subseteq Q$, i.e. $a \in Q$; $c(P) \subseteq Q$ has been established and hence $c(P) = Q$ because both $c(P)$ and Q are minimal primes. The function is continuous. For let $a \in L$. Then, $c^{\leftarrow}(\{Q \in Min(L) : a \notin P\}) = \{P \in Min(\mathcal{J}(L)) : a \notin c(P)\} = \{P \in Min(\mathcal{J}(L)) : (a] \notin P\}$, which means that the inverse image of a basic open set in $Min(L)$ is a basic open set in $Min(\mathcal{J}(L))$. Thus, $Min(L)$ is the continuous image of $Min(\mathcal{J}(L))$ and so is compact due to 1.4. Because of 1.2, L is quasicomplemented.

2.2. Theorem. Let L be a 0-distributive lattice. Then, L is quasicomplemented and each minimal prime ideal of L is the contraction of a unique minimal prime ideal of $J(L)$ if and only for each $J \in J(L)$, there exists $y \in L$ such that $J^{**} = (y]^*$.

Proof. Suppose L is quasicomplemented and if $Q \in Min(L)$, $P_1, P_2 \in Min(J(L))$ are such that $Q = c(P_1) = c(P_2)$ then $P_1 = P_2$. Then, by 2.1 and its proof, $c : Min(J(L)) \longrightarrow Min(L)$ is a bijection. But, by the proof of 2.1, c is continuous. Hence, c is a homeomorphism since each of $Min(L)$ and $Min(J(L))$ is compact and Hausdorff. Because of 1.3 and 1.4, J^* is of the form $(x]^*$ $(x \in L)$ for each $J \in J(L)$. The quasicomplementation on L then implies $J^{**} = (x]^{**} = (x']^*$, as required.

Suppose L satisfies the condition: for each $J \in J(L)$, there exists $y \in L$ such that $J^{**} = (y]^*$. It is clear that L is quasicomplemented. Let $P_1, P_2 \in Min(J(L))$ be such that $c(P_1) = c(P_2)$. Let $J \in P_1$. As L is 0-distributive, $J^* \in J(L)$, $J \vee J^*$ is dense in $J(L)$, and $J \cap J^* = (0] = J^{**} \cap J^*$. As $J(L)$ is 0-distributive and P_1 is a minimal prime ideal, $J^* \notin P_1$ and so $J^{**} \in P_1$. Choose $x \in L$ such that $J^{***} = J^* = (x]^*$. Then $(x]^{**} = J^{**} \in P_1$; so $x \in (x]^{**} \subseteq c(P_1)$. Hence $x \in c(P_2)$. Then, we must have $x \in K$ for some $K \in P_2$, whence $(x] \subseteq K \in P_2$ and $(x] \in P_2$. As P_2 is a minimal prime ideal, $(x]^* \notin P_2$ and so $(x]^{**} \in P_2$. But $J \subseteq J^{**} = (x]^{**}$, and so $J \in P_2$. Thus $P_1 \subseteq P_2$. Because of the minimality of P_2, we conclude that $P_1 = P_2$. Because of the proof of 2.1, each $Q \in \in Min(J(L))$ is the contraction of some $P \in Min(J(L))$ and thus Q is the contraction of a unique $P \in Min(J(L))$.

As a consequence of the proofs of 2.1 and 2.2, together with 1.2, 1.3 and 1.4 we obtain

2.3. Theorem. The following conditions are equivalent for a 0-distributive lattice L .

(a) $Min(L)$ is compact, Hausdorff and extremally disconnected.

(b) L and its lattice of ideals $J(L)$ have homeomorphic spaces of minimal prime ideals.

(c) For each ideal J in L, there is $y \in L$ such that $J^{**} = (y]^*$.

3. Distributive lattices.

3.1. Lemma. Let L be a distributive lattice with 0 and at least one dense element. Then L is quasicomplemented if and only if for each minimal prime ideal P in L and each $x \in L \setminus P$, there exist a dense element d and an element $p \in P$ such that $x \vee p = d \vee p$.

Proof. Let D be the non-empty filter of dense elements in L .

Suppose L is quasicomplemented with $x \in L \setminus P$ for some given minimal prime ideal P . Choose $x' \in L$ such that $x' \wedge x = 0$ and $x \vee x'$ is dense. As P is prime, $x' \in P$. Then, $x \vee p = d \vee p$ with $d = x \vee x' \in D$ and $p = x' \in P$.

Conversely, suppose L satisfies the condition in the lemma. Suppose Q is a prime ideal disjoint from D. Then Q contains a minimal prime P. If Q is not a minimal prime then there is an element $x \in Q \setminus P$. Thus there exist $d \in D$ and $p \in P$ such that $x \vee p = d \vee p$. Then $d \in Q$, an impossibility. Hence any prime ideal Q which is disjoint from D, is a minimal prime. It follows from Stone's theorem that each ideal which is disjoint from the filter D, is contained in a minimal prime ideal. From [6, Proposition 3.4], L is quasicomplemented.

If J is any ideal in a distributive lattice L then it is well-known that the relation $\theta(J)$, given by $a \equiv b(\theta(J))$ $(a, b \in L)$ if and only if $a \vee x = b \vee x$ for some $x \in J$, is a congruence. It is, in fact, the smallest congruence on L having J as a congruence class. When J is prime, the quotient lattice $L/_{\theta(J)}$ is dense, i.e. each non-zero element is dense. We say that a dense element d in $L/_{\theta(J)}$ can be <u>lifted</u> to a dense element x in L if the congruence class of x modulo $\theta(J)$ is d. Lemma 3.1 and these remarks yield the following theorem.

3.2. <u>Theorem</u>. Let L be a distributive lattice with 0 and at least one dense element. Then L is quasicomplemented if and only if, for each minimal prime ideal P in L, each dense element in $L/_{\theta(P)}$ can be lifted to a dense element in L.

References

[1] M. HENRIKSEN and M. JERISON: The space of minimal prime ideals of a commutative ring, Trans.Amer.Math.Soc. 115(1965),110-130.

[2] K.H. HOFMANN and K. KEIMEL: A general character theory for partially ordered sets and lattices, Memoirs Amer.Math.Soc.No.122(1972).

[3] T. KATRIŇÁK: Notes of Stone lattices I (in Russian), Mat.-fyz.časop.16(1966),128-142.

[4] K. KEIMEL: A unified theory of minimal prime ideals, Acta Math.Acad.Sci.Hungar.23(1972),51-69.

[5] J. KIST: Minimal prime ideals in commutative semigroups, Proc.London Math.Soc.(3)13(1963),31-50.

[6] T.P. SPEED: Some remarks on a class of distributive lattices, J.Austral.Math.Soc.9(1969),289-296.

[7] T.P. SPEED: A note on commutative semigroups, J.Austral. Math.Soc.8(1968),731-736.

[8] T.P. SPEED: A note on commutative semigroups II, J.London Math.Soc.(2)2(1970),80-82.

[9] J.C. VARLET: A generalization of the notion of pseudo-complementedness, Bull.Soc.Roy.Sc.Liège 37(1968), 149-158.

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