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RINGS ON COMPLETELY DECOMPOSABLE TORSION-FREE ABELIAN GROUPS

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Abstract: The absolute annihilator $G(*)$ of a completely decomposable torsion free abelian group $G(*)$ is characterized. A chain

$0 \le G(*) = G(1) \le \ldots \le G(\alpha) \le G(\alpha+1) \le \ldots \le G(\mu) = G(\mu+1)$

of "iterated absolute annihilators" of G is then defined. All subgroups $G(\alpha)$ are ideals in every ring on G and when $G = G(\mu)$, some information is obtained about the kinds of ring multiplication which G admits.

Key words: Completely decomposable, absolute annihilator.

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Introduction. Szele [7] defined the nil-degree (Nilstufe) of an abelian group G as the largest integer n such that there is an associative ring R on G with $R^n \neq 0$, if such an n exists. Analogously, we define the strong nil-degree as the largest integer n (if there is one) for which G supports a (not necessarily associative) ring R with $\vec{R}^n$, the subring generated by all products $(\ldots((a_1 a_2)a_3)\ldots)a_n$, non-zero. (The ostensible asymmetry of this definition can be removed by consideration of opposite rings.)

In this note we characterize the absolute annihilator of a completely decomposable abelian group G : the set of elements common to the annihilators of all rings R on G . This leads to the construction of an ascending chain of "absolute ideals" which provides: (i) a sufficient (but far from necessary) condition for G to admit only T -nilpotent ring multiplications; (ii) in some circumstances, an upper bound for the nil-degree of G ; (iii) in all cases, the exact value of the strong nil-degree of G .

We denote the type of a group element x or a rational group X by $T(x)$, $T(X)$ respectively and otherwise follow the conventions of [2]. All groups considered are torsion-free abelian and in the absence of any qualification, rings are associative. A group is nil [6] (resp. strongly nil [5]) if $R^2 = 0$ for every ring (resp. every not necessarily associative ring) R on G . Other notation: G^0 is the zeroring on a group G, R^+ the additive group of a ring R , $\lhd$ indicates an ideal.

1. Completely decomposable nil groups

Ree and Wisner [5] have given a description of the completely decomposable torsion-free nil groups. We begin with a paraphrase of their results, together with a proof, which will be useful later.

Theorem 1.1. Let $G = \bigoplus_{i \in I} X_i$ be a direct sum of rational groups. The following conditions are equivalent.

(i) G is strongly nil.

(ii) G is nil.

(iii) $T(X_i)\,T(X_j) \nleq T(X_k)$ for all $i, j, k \in I$.

Proof. Clearly (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): When considering a rational group X, we lose no generality by assuming that X contains the integers and 1 has any pre-assigned characteristic of appropriate type. Thus, supposing $T(X_i)T(X_j) \leq T(X_k)$ for some $i, j, k \in I$, we may write $X_i = X_i e_i$, $X_j = X_j e_j$, $X_k = X_k e_k$, where $\chi(e_i)\,\chi(e_j) \leq \chi(e_k)$. A multiplication on $X_i \oplus X_j \oplus X_k$ is completely determined by its effect on $\{e_i, e_j, e_k\}$. There are three cases to consider.

(a) If $X_i = X_j = X_k = X = Xe$, then $T(X)$ is idempotent and we can define $e^2 = e$.

(b) If $X_i = X_j = X = Xe \neq X_k$, we can use the multiplication table

	e	e_k
e	e_k	0
e_k	0	0

If $X_j = X_k = X = Xe \neq X_i$, then $T(X) \leq T(X_i)T(X) \leq T(X)$, so $T(X_i)T(X_i) \leq T(X_i)T(X) = T(X)$, and we are back to the previous case.

(c) If X_i, X_j and X_k are all distinct, the following table can be used:

	e_i	e_j	e_k
e_i	0	e_k	0
e_j	e_k	0	0
e_k	0	0	0

.

In every case we have defined an associative ring R on $X_i \oplus X_j \oplus X_k$ and $R^2 \neq 0$. Thus $G = \left(R \oplus \left[\bigoplus_{\ell \neq i,j,k} X_\ell\right]^0\right)^+$ is not nil.

(iii) $\Longrightarrow$ (i): If R is a ring (not necessarily associative) on G with $R^2 \neq 0$, then $X_i X_j \neq 0$ for some $i, j \in I$. Let $x \in X_i$, $y \in X_j$ be such that

$$0 \neq xy = z_1 + \dots + z_m \,, \; 0 \neq z_r \in X_{i_r} \,.$$

Then

$$T(X_i)\,T(X_j) \leq T(xy) = T(X_{i_1}) \cap \dots \cap T(X_{i_r}) \leq T(X_{i_1}) \,.$$

__Remark.__ The second possibility mentioned in (b) of the proof just given can occur even for non-idempotent types, e.g. the types corresponding to the characteristics

$$\chi_1 = (0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots)$$

$$\chi_2 = (1, \infty, 1, \infty, 1, \infty, 1, \infty, 1, \infty, 1, \infty, \dots) \,.$$

The assumption that it cannot leads to some incorrect statements in [1]. In particular, while it is true that the direct sum of two nil rational groups has nil-degree 1 or 2, the strong nil-degree need not be defined. For example if $X = Xe$ and $Y = Yf$ are rational with $\chi(e) = \chi_1$ and $\chi(f) = \chi_2$, consider the non-associative ring defined.

on $X \oplus Y$ by the multiplication table

	e	f
e	0	f
f	f	0

2. <u>The absolute annihilator</u>

Fuchs ([2], Problem 94) refers to the absolute annihilator of a group G, the set of elements belonging to the annihilator of every ring on G. In this section we investigate the absolute annihilator, which we denote by $G(*)$, when G is completely decomposable.

<u>Theorem 2.1</u>. Let $G = \bigoplus_{i \in I} X_i$ be a direct sum of rational groups. Then

$$G(*) = \bigoplus \{X_i \mid i \in I \text{ and } \exists \text{ no } j, k \in I \text{ with } T(X_i)T(X_j) \leq T(X_k)\} .$$

<u>Proof</u>. Note firstly that if $X_i \cap G(*) \neq 0$, then $X_i \subseteq G(*)$.

If $T(X_i)T(X_j) \leq T(X_k)$ for some $j, k \in I$, then as in the proof of Theorem 1.1, there is a ring A on G with $X_i A \neq 0$, so that $X_i \nsubseteq G(*)$. Conversely, if $X_i \nsubseteq G(*)$ then for any non-zero $x \in X_i$, there exists a ring R on G in which $x X_j \neq 0$ for some $j \in I$. If $xy \neq 0$, where $y \in X_j$, let

$$xy = z_1 + \dots + z_m$$

where z_k is a non-zero element of X_{i_k} for (distinct)

$i_1, \dots, i_m \in I$. Then

$$T(X_i)\,T(X_j) \leq T(xy) = T(x_1) \cap \dots \cap T(x_m) \leq T(x_m) = T(X_{i_m}).$$

Finally, let $x_1 + \dots + x_m$ be any element of $G(*)$, where $0 \neq x_r \in X_{i_r}$ for $r = 1, \dots, m$ and the i_r are distinct. We complete the proof by showing that $x_1 \in G(*)$. If $x_1 \notin G(*)$, then $T(X_{i_1})\,T(X_j) \leq$ $\leq T(X_k)$ for some $j, k \in I$ and as in Theorem 1.1 we can define a commutative ring A on G such that for some $\ell \in I$, $X_{i_1} X_\ell \neq 0$ but $X_m X_\ell = 0$ for all other $m \in I$. But then for $0 \neq y \in X_\ell$ we have

$$x_1 y = (x_1 + \dots + x_m)\, y = 0 .$$

We now consider a chain

$$0 \subseteq G(1) \subseteq G(2) \subseteq \dots \subseteq G(\alpha) \subseteq \dots$$

of subgroups of G, defined inductively as follows:

$$G(1) = G(*)\,;\; G(\alpha+1)/G(\alpha) = [G/G(\alpha)](*)\,;$$
$$G(\beta) = \bigcup_{\alpha<\beta} G(\alpha) \quad \text{if } \beta \text{ is a limit ordinal.}$$

Clearly $G(\mu) = G(\mu+1)$ for some ordinal μ.

A straightforward transfinite induction argument provides a proof of

<u>Lemma 2.2</u>. Let $G = \bigoplus_{i\in I} X_i$ be a direct sum of rational groups. For every α, there exists a subset I_α of I such that $G(\alpha) = \bigoplus_{i \in I_\alpha} X_i$.

<u>Theorem 2.3.</u> Let $G = \bigoplus_{i\in I} X_i$ be a direct sum of rational groups, R a ring on G. Then $G(\alpha) \lhd R$ for eve-

ry α .

<u>Proof</u>. Let f be an endomorphism of G, $x \in X_i \subseteq G(*)$. Then $f(x) = 0$ or

$$0 \neq f(x) = x_1 + \dots + x_m$$

where $x_r \in X_{i_r}$, $r = 1,\dots,m$ and the i_r are distinct. If $f(x) \notin G(*)$, then some $x_r \notin G(*)$, so $T(X_{i_r})T(X_j) \not\leq T(X_k)$ for some $j, k \in I$. It follows that $T(X_i)\,T(X_j) \not\leq T(X_k)$, which is impossible, so $f(X_i) \subseteq G(*)$. Hence $G(*)$ is fully invariant, so that $G(1) = G(*) \lhd R$. If now $G(\alpha) \lhd R$, then Lemma 2.2 implies that $[R/G(\alpha)](*) \lhd R/G(\alpha)$, and thus $G(\alpha+1) \lhd R$. At limit ordinals the result is clear.

Suppose now that $G(\mu) = G$ for some ordinal μ . In any ring R on G , $G(\alpha+1)/G(\alpha) \subseteq (0 : R/G(\alpha))$ i.e. $G(\alpha+1)R \subseteq G(\alpha)$ and $RG(\alpha+1) \subseteq G(\alpha)$. Thus

$$0 \subseteq G(1) \subseteq G(2) \subseteq \dots \subseteq G(\alpha) \subseteq \dots \subseteq G(\mu) = R$$

is a two-sided annihilator series for R in the sense of [3]. Thus by Theorem 1.6 of [3] and § 3 of [4], we have

<u>Corollary 2.4</u>. If G is as in Theorem 2.3 and $G = G(\mu)$ for some ordinal μ , then any ring R on G is left and right T-nilpotent. If in addition μ is finite, then $R^{\mu+1} = 0$ for any such R .

We conclude this section with an "internal" characterization of the subgroups $G(n)$ for finite n. A π-matrix is a $2\times m$ matrix

$$\begin{bmatrix} \tau_{11} & \tau_{12} \dots & \tau_{1m} \\ \tau_{21} & \tau_{22} \dots & \tau_{2m} \end{bmatrix}$$

of types such that $\tau_{1i}\ \tau_{2i} \leq \tau_{1,i+1}$ for $i = 1, 2, \dots, m-1$.

Proposition 2.5. Let $G = \bigoplus_{i \in I} X_i$ be a direct sum of rational groups. Then

$$G(m) = \oplus\{X_j \mid \exists \text{ no } 2\times(m+1)\ \pi\text{-matrix over } \{T(X_i) \mid i \in I\} \text{ with } \tau_{11} = T(X_j)\}.$$

Proof. The result is true for $m = 1$ (Theorem 2.1); if it is true for m, let

$$G = G(m) \oplus H = G(m) \oplus H(*) \oplus K = G(m+1) \oplus K .$$

If there is a $2\times(m+2)$ π-matrix

$$\begin{bmatrix} T(X_i) & T(X_k) \dots & \tau_{1,m+2} \\ T(X_j) & \tau_{22} \quad \dots & \tau_{2,m+2} \end{bmatrix}$$

with $X_i \subseteq G(m+1)$, then $X_i \nsubseteq G(m)$ (strike out the last column) and similarly $X_j \nsubseteq G(m)$. Thus $X_i \subseteq H(*)$ and $X_j \subseteq H$. But then $X_k \nsubseteq H$, so $X_k \subseteq G(m)$, which is impossible, as $T(X_k)$ is the $(1,1)$ entry in a $2\times(m+1)$ π-matrix. Conversely, if X_i satisfies the condition for $2\times(m+2)$ π-matrices, we need only look at the case where $X_i \subseteq H$. If $X_i \subseteq K$, then $T(X_i)\ T(X_j) \leq T(X_k)$ for some $X_j, X_k \subseteq H$. But then there is a $2\times(m+1)$ π-matrix $M = [\tau_{ij}]$ with $\tau_{11} = T(X_k)$ which can be augmented to a $2\times(m+2)$ π-matrix

$$\begin{bmatrix} TX_i & \\ & M \\ TX_j & \end{bmatrix}.$$

Hence we conclude that $X_i \subseteq H(*) \subseteq G(m+1)$.

3. Some examples

A completely decomposable group G need not have an absolute annihilator series in order to admit only T-nilpotent multiplications. For instance if G is the group in the Remark in § 1, then $G(1) = 0$, but $R^3 = 0$ for every ring R on G .

Even when $G = G(m) \neq G(m+1)$, G can have nil-degree $\leq m$: Let $G = X_1 \oplus X_2 \oplus X_3 \oplus X_4 \oplus X_5$, where the types of $X_1, \dots, X_5$ are those of the characteristics

$$\begin{array}{l}(1,\ 0,\ 0,\ 1,\ 0,\ 0, \dots)\\(0,\ 1,\ 0,\ 0,\ 1,\ 0, \dots)\\(1,\ 1,\ 0,\ 1,\ 1,\ 0, \dots)\\(0,\ 0,\ 1,\ 0,\ 0,\ 1, \dots)\\(1,\ 1,\ 1,\ 1,\ 1,\ 1, \dots)\end{array}$$

respectively. It is routine to verify that $R^3 = 0$ for any ring R on G but that $\begin{bmatrix} T(X_1) & T(X_3) & T(X_5) \\ T(X_2) & T(X_4) & T(X_5) \end{bmatrix}$ is a π-matrix.

Any direct sum of finitely many rational groups with non-idempotent types has a nil-degree (see e.g.[8], Theorem 3.1). With infinite rank the situation can be quite different. Consider the characteristics

$(\infty,\ m,\ \infty,\ m,\ \infty,\ m,\ \infty,\ m,\ \infty,\ m, \dots)$

$(\infty,\ 0,\ m,\ 0,\ \infty,\ 0,\ m,\ 0,\ \infty,\ 0, \dots)$

$(\infty,\ 0,\ 0,\ 0,\ m,\ 0,\ 0,\ 0,\ \infty,\ 0, \dots)$

$\dots;$

$m = 1, 2, \dots$. They form a semigroup $(\{\chi_i \mid i \in I\}, \cdot)$. Let $X_i = X_i e_i$ be a rational group containing e_i with $\chi(e_i) = \chi_i$ and write $\chi_{ij} = \chi_i \chi_j$ etc. Then a ring R is defined on $\bigoplus_{i \in I} X_i$ by the multiplication rule $e_i e_j = e_{ij}$. Since for any i one can find j with $\chi_j \chi_i = \chi_i$, we have $x e_i = (x e_i) e_j$ and R is idempotent and since $(\sum_{i=1}^{m} x_i e_i)(\sum_{j=1}^{m} y_j e_j) = \sum_{i,j=1}^{m,m} x_i y_i e_{ij}$, R has no zero-divisors.

4. Non-associative rings

The absolute annihilator series furnishes more precise information about the non-associative rings which can be defined on a completely decomposable group G . If $G = G(m)$ (m finite) it is easily proved that $\vec{R}^{s} \le G(m-s+1)$ for $s = 1, \dots, m$, whence $\vec{R}^{m+1} = 0$ for any ring R on G .

<u>Proposition 4.1</u>. Let $G = \bigoplus_{i \in I} X_i$ be a direct sum of rational groups. There is a (not necessarily associative) ring R on G with $\vec{R}^{m} \neq 0$ if and only if there is a $2 \times m$ π -matrix over $\{T(X_i) \mid i \in I\}$.

<u>Proof</u>. If there is such a matrix $[T(X_{ij})]$, let $X_{ij} = X_{ij} e_{ij}$ where the characteristics $\chi(e_{ij})$ sa-

tisfy the relations the matrix requires of their types. Define $e_{1i}e_{2i} = e_{i,i+1}$ and let all products not thus accounted for be zero. (Note that e_{ij} and e_{rs} can be equal for different (i,j) and (r,s) .) Then $(\ldots(((e_{11}e_{21})e_{22})e_{23})\ldots)e_{2,m-1} = e_{1m} \neq 0$. On the other hand, if there is no such $2\times m$ matrix, then $G = G(m-1)$ and $\vec{R}^{m} = 0$ for all rings R on G .

Summarizing, therefore, we have

Theorem 4.2. Let $G = \bigoplus_{i\in I} X_i$ be a direct sum of rational groups. The following conditions are equivalent:

(i) $G = G(m)$, $m < \infty$ and $G \neq G(m-1)$.

(ii) There are $2\times m$, but no $2\times(m+1)$ π-matrices over $\{T(X_i) \mid i \in I\}$.

(iii) G has strong nil-degree m .

R e f e r e n c e s

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