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BOUNDARY VALUE AND PERIODIC PROBLEM FOR THE EQUATION $x''(t) + g(x(t)) = p(t)$

(Preliminary communication)

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Abstract: Under the assumption g is a continuous function with $\lim_{|\xi| \to +\infty} \frac{g(\xi)}{\xi} = +\infty$ it is considered the boundary value and periodic problem for the equation $x''(t) + g(x(t)) = p(t)$.

The periodic problem is also investigated in the cases of various growths of the function g.

Key words: Boundary value problem, periodic problem, weak solution, classical solution, nonlinear ordinary differential equation of the second order.

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Let g be a continuous real valued function defined on the whole real line R. We shall consider the nonlinear differential equation

$$x''(t) + g(x(t)) = p(t) \quad \left('' = \frac{d^2}{dt^2}\right), \tag{1}$$

where p is a given right hand side. If a, b, c, d are the real numbers we shall consider the boundary value problem

$$a\,x(0) + b\,x'(0) = 0 = c\,x(1) + d\,x'(1). \tag{2}$$

A solution of (1) is said to satisfy the periodic problem if

(3) $$x(0) = x(1) \;, \quad x'(0) = x'(1) \;.$$

Definition. Let $p \in L_1(0,1)$. A continuously differentiable function on $\langle 0,1\rangle$ is said to be a weak solution of the equation (1) if for any $t \in \langle 0,1\rangle$ it holds

$$x(t) = x(0) + t x'(0) + \int_0^t (t-s)(p(s) - g(x(s)))\,ds \;.$$

It is easy to see that if p is continuous then the weak solution x of (1) has second derivative which is continuous on $\langle 0,1\rangle$ and the equation (1) is satisfied in each $t \in \langle 0,1\rangle$.

The existence of a weak solution (or classical solution) of the problem (1), (2) under the assumption:

there exist $\alpha \geqq 0$, $\beta \geqq 0$ such that $|g(\xi)| \leqq \alpha + \beta|\xi|$, follows immediately from the abstract results given in [3 - 5, 8, 9].

Interesting nonlinearity of the function g is considered in [1]. Many papers deal with the problem (1), (3). But by our meaning, the case of a special right hand side is considered (see e.g. [2]), or there are supposed some additional assumptions ([6, 7]).

Our results may be formulated as follows:

Theorem 1. Let

$$\lim_{|\xi| \to +\infty} \frac{g(\xi)}{\xi} = +\infty$$

Then the boundary value problem (1), (2) has for each $p \in L_1(0,1)$ an infinite number of distinct weak solutions.

Theorem 2. Let the function g satisfy the assumptions of Theorem 1. Then for any right hand side $p \in L_1(0,1)$ the periodic problem (1), (3) has at least one weak solution. (The proofs of Theorems 1 and 2 use essentially the shooting method and the comparison theorems.)

Theorem 3. Let g be a bounded continuous function on R. Suppose that there exist

$$\lim_{\xi \to +\infty} g(\xi) = g(+\infty) \quad ,$$
$$\lim_{\xi \to -\infty} g(\xi) = g(-\infty) \quad .$$

If the inequalities $g(-\infty) < g(\xi) < g(+\infty)$ hold for each $\xi \in R$ then the periodic problem (1), (3) has a weak solution for the right hand side $p \in L_1(0,1)$ if and only if

(4) $$g(-\infty) < \int_0^1 p(t)\,dt < g(+\infty) \ .$$

Theorem 4. Under the assumptions of Theorem 3 the condition (4) is necessary and sufficient to be the periodic problem solvable (in the classical sense) for a continuous right hand side p .

Theorem 5. Let g be an odd continuous monotone function on R and

$$\lim_{\xi \to +\infty} g(\xi) = +\infty \quad .$$

Suppose that there exist $\alpha \geqq 0$ and $\beta \in (0, 9^{-1})$ such that $|g(\xi)| \leqq \alpha + \beta|\xi|$ for each $\xi \in R$. Then for arbitrary $h \in L_1(0,1)$ there exists at least one weak solution of the periodic problem (1), (3).

(The proofs of Theorems 3 - 5 are based on the abstract method for the solvability of the nonlinear equations (see [3]).)

The detailed proofs of the results will be presented in a paper to be published later in Czech.Math.Journal or Čas.Pěst.Mat. where further comments and references will be given.

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