Commentationes Mathematicae Universitatis Carolinae

15,2 (1974)

# A MIXED FINITE ELEMENT METHOD CLOSE TO THE EQUILIBRIUM MODEL

(Preliminary communication)

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Abstract: A new type of mixed finite element method is derived which gives approximate solutions to the Dirichlet Boundary value problems for one elliptic equation and for the system of equations, governing the plane elasticity. The Galerkin approximations are vector functions converging to the chosen components of the cogradient (or of the stress tensor) and to the solution itself.

Key words: finite elements.

AMS: 65N30 Ref. Ž. 8.33

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## 1. Variational formulation and a mixed finite element model

Let us consider the following problem

$$(1.1) \qquad -a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = f \quad \text{in } \Omega ,$$

$$u = 0 \quad \text{on } \Gamma ,$$

where $f \in L_2(\Omega)$, $a_{ij} = const$ form a symmetric positive definite matrix $n \times n$ .

Let us introduce

$$\mathcal{H} = [W^{1,2}(\Omega)]^n , \qquad \|\lambda\|_{\mathcal{H}} = (\sum_{i=1}^{n} \|\lambda_i\|_1^2)^{\frac{1}{2}}$$

and the bilinear form $B(\lambda, \mu)$ on $\mathcal{H} \times \mathcal{H}$ as follows:

$$(1.2) \quad B(\lambda, \mu) = (b_{ij}\lambda_i, \mu_j) - \gamma^{-1}(div\,\lambda - \alpha_j\lambda_j, div\,\mu - \alpha_j\mu_j) ,$$

where $b_{ij}$ are entries of the matrix inverse to $(a_{ij})$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a constant non-zero vector,

$\gamma = a_{ij}\alpha_i\alpha_j$, $\quad div\,\lambda = \dfrac{\partial \lambda_j}{\partial x_j}$ .

We have shown in [1] the connection between the solution of (1.1) and the solution of the following problem: to find $\hat{\lambda} \in \mathcal{H}$ such that

$$B(\hat{\lambda}, \mu) = \gamma^{-1}(f, div\,\mu - \alpha_j \mu_j) \qquad \forall \mu \in \mathcal{H} \; . \tag{1.3}$$

The connection is given by the

**Theorem 1.** Let the weak solution $u$ of (1.1) belong to $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for any $f \in L_2(\Omega)$ and $\|u\|_2 \le C\|f\|$ .

Then (1.3) has precisely one solution $\hat{\lambda} \in \mathcal{H}$ and it holds

$$\hat{\lambda}_i = a_{ij}\left(\frac{\partial u}{\partial x_j} + \alpha_j u\right), \quad i = 1, 2, \dots, n \; , \tag{1.4}$$

$$u = \gamma^{-1}(\alpha_j \hat{\lambda}_j - div\,\hat{\lambda} - f) \; , \tag{1.4'}$$

$$\|\lambda\|_{\mathcal{H}} \le c\,\|f\| \; , \tag{1.5}$$

where $c$ is a constant independent of $f$ .

On the basis of (1.3) the Galerkin approximations can be defined. For simplicity let us restrict ourselves only to the following model problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \subset E_n \; , \\ u &= 0 \quad \text{on } \Gamma \; . \end{aligned} \tag{1.6}$$

Let us set

$$\begin{aligned} &\alpha_1 = \alpha_2 = \dots = \alpha_{n-1} = 0 \, , \\ &\alpha_n = \alpha_0 h^{-1-\varepsilon} \; (h \in (0,1) , \; \alpha_0 > 0 , \; \varepsilon > 0 ) . \end{aligned} \tag{1.7}$$

Introducing

$$\bar{\lambda}_m = \alpha^{-1}\lambda_m \; , \quad \bar{\mu}_m = -\alpha^{-1}\mu_m$$

into $B(\lambda, \mu)$ we obtain another bilinear form

(1.8) $\tilde{B}(\lambda_1,\dots,\lambda_{m-1},\bar{\lambda}_m;\mu_1,\dots,\mu_{m-1},\bar{\mu}_m) = B(\lambda_1,\dots,\alpha\bar{\lambda}_m;\mu_1,\dots,-\alpha\bar{\mu}_m)$.

It is readily seen that the problem to find $\lambda \in \mathcal{H}$ such that

(1.9) $\tilde{B}(\bar{\lambda},\bar{\mu}) = \alpha^{-2}(f, \sum_1^{m-1}\mu_{j,j} - \alpha\bar{\mu}_{m,m} + \alpha^2\mu_m) \quad \forall \bar{\mu} \in \mathcal{H}$

is equivalent to (1.3).

In order to define Galerkin approximations, we introduce two families of finite-dimensional subspaces $V_h$, $V_{h_m}$ $(0 < h \le 1,\ 0 < h_m \le 1)$ which satisfy the following assumptions:

(i) (Conformity) $V_h \subset W^{1,2}(\Omega)$, $V_{h_m} \subset W_0^{1,2}(\Omega)$,

(ii) (Approximability) $\exists\, \varkappa \ge 2$, $\forall w \in W^{\varkappa,2}(\Omega)\ \exists \chi \in V_h$ :

$$\|w - \chi\|_j \le C h^{\varkappa - j}\|w\|_\varkappa \; ;$$

$\exists\, \varkappa_m \ge 2$, $\forall v \in W^{\varkappa_m,2}(\Omega) \cap W_0^{1,2}(\Omega)\ \exists \psi \in V_{h_m}$ :

$$\|v - \psi\|_j \le C h^{\varkappa_m - j}\|v\|_{\varkappa_m} \, , \qquad (j = 0,1) \, ,$$

(iii) (Inverse inequality) A constant $C$ exists, independent of $\chi$ and $h$ such that for sufficiently small $h$ and any $\chi \in V_h$

$$\|\chi\|_1 \le C h^{-1}\|\chi\| \; .$$

Denote $\quad V(h, h_m) = (V_h)^{n-1} \times V_{h_m} \subset \mathcal{H}$ .

We say that an element $\bar{\lambda}^h \in V(h, h_m)$ is a Galerkin approximation to the solution $\bar{\hat{\lambda}} = (\hat{\lambda}_1, \dots, \bar{\hat{\lambda}}_m)$ of the problem (1.9) if

(1.10) $$\tilde{B}(\bar{\lambda}^h, \bar{\mu}) = \alpha^{-2}(f, \sum_1^{m-1} \mu_{j,j} - \alpha\bar{\mu}_{m,m} + \alpha^2\bar{\mu}_m) \quad \forall \bar{\mu} \in V(h, h_m).$$

The following rate of convergence is proved for the Galerkin approximations

Theorem 2. Let the solution $u$ of the problem (1.6) belong to $W^{q,2}(\Omega)$ where $q \geq max(\varkappa + 1, \varkappa_m)$ and to $W^{2,2}(\Omega)$ for any $f \in L_2(\Omega)$.

Then for sufficiently small $h$ the Galerkin approximations are defined uniquely by (1.10) and it holds

$$\sum_{j=1}^{m-1} \| \frac{\partial u}{\partial x_j} - \lambda_j^h \| + \| \frac{\partial u}{\partial x_m} - \frac{\partial \bar{\lambda}_m^h}{\partial x_m} \| + \| u - \bar{\lambda}_m^h \| \leq$$

$$\leq C [ h^{\varkappa-1} + h^{\varepsilon} + h^{-1} h_m^{\varkappa_m} + h_m^{\varkappa_m - 1} ] \| u \|_q .$$

## 2. A mixed finite element method for plane elasticity

As a model problem of elliptic systems the Dirichlet problem for linear plane elasticity was chosen [2], i.e.

(2.1) $$(\lambda_0 + \mu_0) u_{j,ji} + \mu_0 \mu_{i,jj} + F_i = 0 \qquad \text{in } \Omega \subset E_2 ,$$

$$u_i = 0 \quad \text{on } \Gamma , \qquad (i = 1, 2) ,$$

where $w_{j} = \frac{\partial w}{\partial x_j}$, $\lambda_0 > 0$ and $\mu_0 > 0$ are Lamé's constants and $F_i$ the body force components. A repeared index implies summation over 1,2.

$\Omega$ is a bounded domain with Lipschitz boundary $\Gamma$. Applying the same idea as in Section 1, we define:

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) ,$$

and transforming $\lambda_1, \lambda_2$ into $\bar\lambda_1, \bar\lambda_2$ properly, the Galerkin approximations $\bar\lambda^h$ can be defined on the basis of a new variational formulation of (2.1) similarly to those of Section 1 and an error estimate analogous to that of Theorem 2 holds. As a result, $\lambda_3^h$ converge to the shear stress in $L_2(\Omega)$, whereas $\bar\lambda_j^h$ and $\bar\lambda_{j,j}^h$ converge in $L_2(\Omega)$ to $u_j$ and $u_{j,j}$, respectively.

3. Application of curved elements

In [3] the application of two types of curved elements is shown, namely (i) of elements which describe the boundary arcs exactly, analyzed by Zlámal [5] and (ii) of elements, interpolating the boundary arcs, the theory of which was given e.g. by Ciarlet and Raviart [6].

We studied the convergence of Galerkin approximations to the solution of the model problem (1.6) in $E_2$ with $\Gamma \in C^\infty$.

Using (ii) for the case that the boundary is approximated piecewise by parabolas and the basic space of functions on the reference element consists of quadratic polynomials we obtain the rate of convergence of the Galerkin approximations as follows in

Theorem 4. Let the solution $u$ of (3.1) belong to $W^{p,2}(\Omega)$, $p \ge 3$. Then

$$\left\| \frac{\partial u}{\partial x_1} - \lambda_1^h \right\|_{\Omega_h} + \left\| \frac{\partial u}{\partial x_2} - \frac{\partial \bar\lambda_2^h}{\partial x_2} \right\|_{\Omega_h} + \| u - \bar\lambda_2^h \|_{\Omega_h} \le$$
$$\le C\,[\, h^{\sigma} \| u \|_{p,\Omega} + h^{\sigma} ( \| u \|_{3,\Omega} + \| f \|_{\Omega} ) ] ,$$

where $\delta = min(1, \varepsilon)$ for $p = 3$, $\delta = min(2, \varepsilon)$ for $p = 4$, $\gamma = min(1 + \varepsilon, 2)$ and $\Omega_h$ is the region bounded by the parabolas.

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(Oblatum 29.4.1974)