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COMPLETELY ADDITIVE DISJOINT SYSTEM OF BAIRE SETS IS OF  
BOUNDED CLASS

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**Abstract:** The theorem in the title is proved; this result corrects the proof of Lemma 2 in [F], and thus makes all deep results in [F] verified.

**Key words:** Completely additive systems, Baire classification of sets.

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1. Frolík [F] claimed to prove several deep results on completely metrizable spaces concerning Baire measurable maps and maps of bounded class. His proofs are based on Hensel's lemma (see [F]) and on its converse (see Lemma 2 [F, p.140]) which gives a characterization of disjoint Baire completely additive systems. But the proof of Lemma 2 in [F] does not seem to be correct. (Why should the sets  $X'_n$  be closed in  $X'$ ?). To give a correct proof of this lemma it would be sufficient to answer positively the following question: If  $\{X_\alpha \mid \alpha \in A\}$  is a disjoint family in an absolute Souslin space such that the union of each subfamily of  $\{X_\alpha\}$  is a Baire set, is it true that the family  $\{X_\alpha\}$  ranges in some Baire class?

In this note we prove that the answer is yes even in

more general setting. In spite of what was said before it would follow that the theorems in F hold.

2. Let  $X$  be any set and let  $\mathcal{B}_0$  be any family of subsets of  $X$ . Let  $\mathcal{B} = \{\mathcal{B}_\alpha \mid \alpha < \omega_1\}$  be defined as follows:

$\mathcal{B}_\alpha$  is the collection of all countable unions or intersections of elements of  $\cup \{\mathcal{B}_\beta \mid \beta < \alpha\}$  according to as  $\alpha$  is odd or even.

For  $B \in \mathcal{B}$  put class  $B = \min \{\alpha \mid B \in \mathcal{B}_\alpha\}$ .

As an example we can have  $X$  a topological (or uniform) space and  $\mathcal{B}_0$  the family of zero sets.

Our result can be expressed now as follows.

Theorem. Let  $\{X_\alpha \mid \alpha \in A\}$  be a disjoint family of subsets of  $X$ . If the union of each subfamily of  $\{X_\alpha\}$  belongs to  $\mathcal{B}$  then the family  $\{X_\alpha\}$  ranges in some  $\mathcal{B}_\alpha$ .

3. A limit ordinal number  $\varepsilon$  is called regular if any co-final subset of  $T_\varepsilon$  is of the type  $\varepsilon$ .

Lemma. Let  $\varepsilon$  be a regular ordinal number. Let  $A$  be any set and  $\varphi$  a map of  $\exp A$  into  $T_\varepsilon$ . Suppose that there is a map  $\eta : T_\varepsilon \times T_\varepsilon \rightarrow T_\varepsilon$  such that:

$$\varphi(A_1 \cap A_2) \subseteq \eta(\varphi(A_1), \varphi(A_2)).$$

Then there is an  $\alpha < \varepsilon$  such that  $\varphi(a) \subseteq \alpha$  for any  $a \in A$ .

Proof. Suppose that the family  $\{\varphi(a) \mid a \in A\}$  is

not bounded by any  $\alpha < \varepsilon$ . By induction we can, for any  $\alpha < \varepsilon$ , choose an  $a_\alpha \in A$  such that

$$\varphi(a_\alpha) \geq \alpha, \quad \varphi(a_\alpha) > \sup \{ \varphi(a_\beta) \mid \beta < \alpha \}.$$

(If  $a_\alpha$  have been defined for  $\alpha < \alpha_0 < \omega$ , then  $\sup \{ \varphi(a_\alpha) \mid \alpha < \alpha_0 \} < \varepsilon$  since  $\varepsilon$  is regular.)

Using the correspondence between  $\alpha$  and  $a_\alpha$  we see that we only have to find a contradiction in case  $A = T_\varepsilon$  and  $\varphi(\alpha) \geq \alpha$  for any  $\alpha < \varepsilon$ .

Let  $K_0$  be the set of all limit ordinals  $< \varepsilon$  and let  $K_\alpha$  be the set of all limit members of  $\bigcap_{\beta < \alpha} K_\beta$  in itself ( $\alpha < \varepsilon$ ). For any  $\alpha < \varepsilon$  the set  $\bigcap_{\beta < \alpha} K_\beta$  is a closed subset of  $T_\varepsilon$ . By induction we easily show that this set is also co-final. (If this holds for any  $\alpha < \alpha_0 < \varepsilon$  then, for given  $\gamma < \varepsilon$ , put  $\gamma_\beta = \min(K_\beta \cap (T_\varepsilon \setminus T_\gamma))$ . Since the set  $\{ \gamma_\beta \mid \beta < \alpha_0 \}$  is not co-final, we have

$$\sup \{ \gamma_\beta \mid \beta < \alpha_0 \} \in \bigcap_{\beta < \alpha_0} K_\beta .)$$

Now choose  $\alpha_\beta \in \bigcap_{\gamma < \beta} K_\gamma \setminus K_\beta$  such that

$$\varphi(\alpha_\beta) > \eta(\beta, \varphi(\bigcap_{\gamma < \beta} K_\gamma \setminus K_\beta)).$$

(We put  $\bigcap_{\gamma < 0} K_\gamma = T_\varepsilon$ .)

(The existence of  $\alpha_\beta$  follows from the co-finality of the sets  $\bigcap_{\gamma < \beta} K_\gamma \setminus K_\beta$ .)

$$\begin{aligned} \text{Since } \varphi(\alpha_\beta) &= \varphi(\{ \alpha_\gamma \mid \gamma < \varepsilon \} \cap (\bigcap_{\gamma < \beta} K_\gamma \setminus K_\beta)) \leq \\ &\leq \eta(\varphi(\{ \alpha_\gamma \mid \gamma < \varepsilon \}), \varphi(\bigcap_{\gamma < \beta} K_\gamma \setminus K_\beta)) \end{aligned}$$

we obtain the contradiction putting  $\beta = \varphi(\{ \alpha_\gamma \mid \gamma < \varepsilon \})$ .

Remark. The preceding lemma holds for arbitrary ordinal numbers if we suppose that  $\eta$  is monotone (i.e.  $\eta(\alpha, \beta) \leq \eta(\alpha', \beta')$  if  $\alpha \leq \alpha'$  and  $\beta \leq \beta'$ ). For non-limit ordinals the proof is obvious. To prove it for a limit ordinal  $\varepsilon$  one finds a co-final subset  $T$  of  $T_\varepsilon$  of the smallest possible type and defines  $\bar{\varphi}(B) = \min\{\gamma \in T \mid \varphi(B) \leq \gamma\}$  and  $\bar{\eta}(\alpha, \beta) = \inf\{\gamma \in T \mid \eta(\alpha, \beta) \leq \gamma\}$  for  $\alpha, \beta \in T$ . Using Theorem 2 (where  $T_\varepsilon$  is replaced by  $T$ ) one obtains the proof.

Proof of the Theorem. Put, for  $B \subset A$ ,  $\varphi(B) = \text{class}(\bigcup_{a \in B} X_a)$ . Using the lemma for  $\varepsilon = \omega_1$  (put  $\eta(\alpha, \beta) = \max(\alpha, \beta) + 1$ ) we obtain that the family  $\{\varphi(a) \mid a \in A\}$  is bounded, i.e.  $\{X_a \mid a \in A\}$  ranges in some  $\beta_\alpha$ .

#### R e f e r e n c e

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