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EXISTENCE THEOREM FOR A GENERALIZED HAMMERSTEIN TYPE EQUATION

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Abstract: An existence theorem is obtained for a generalized Hammerstein type equation.

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In [4] Browder has obtained an existence theorem for a generalized Hammerstein type equation

(1) $$u + \sum_{i=1}^{m} A_i F_i u = 0$$

where each A_i is a linear operator from a function space X to its dual space X^* and F_i is a nonlinear operator from X^* to X. Each linear operator A_i is assumed to be angle-bounded and the nonlinear operators $F_1, F_2, \dots, F_m$ satisfy a condition of the type

(2) $$\sum_{i=1}^{m} (F_i(u) - F_i(v), u_i - v_i) \geq -c \sum_{i=1}^{m} \| u_i - v_i \|^2_{X^*}$$

where c is some constant and $u = \sum_{i=1}^{m} u_i$, $v = \sum_{i=1}^{m} v_i$.

Condition (2), though a natural generalization of the monotonicity condition, is rather hard to verify. In this paper we weaken this condition on the operators $F_1, \dots, F_m$ by assuming additional hypothesis of compactness on the linear operators A_i. In the application of this theory to the case where the A_i are integral operators, the assumption of compactness is a natural one.

We now introduce the following definitions: Let X be a real Banach space, X^* its dual and let (w, u) denote the duality pairing between the elements w in X^* and u in X.

Definition 1. A mapping T from X to X^* is said to be of the type (M) if the following conditions hold:

(M_1) - If a sequence $\{u_m\}$ in X converges weakly to an element u in X (written $u_m \rightharpoonup u$), the sequence $Tu_m \rightharpoonup w$ in X^* and $\limsup (Tu_m, u_m) \leq (w, u)$, then $Tu = w$.

(M_2) - T is continuous from finite dimensional subspaces of X to the space X^* endowed with the weak*-topology.

It should be observed that if T is monotone and continuous then T is of type (M) [2]. The concept of mappings of type (M) was first introduced by Brezis [2] using filters and later used by de Figueiredo and Gupta in [5].

Definition 2. If T is a bounded monotone linear map of X into X^*, then T is said to be angle-bounded with constant $a \geq 0$ if for all u, v in X

$$|(Tu, v) - (Tv, u)| \leq 2a\{(Tu, u)\}^{1/2}\{(Tv, v)\}^{1/2} .$$

It is clear that every monotone map T which is symmetric is angle-bounded with $a = 0$. In proving existence theorem we shall appeal to Proposition 3 of [5] and Theorem 4 of [3] which we now state.

Proposition 1 (de Figueiredo and Gupta). Let X be a reflexive Banach space and T be a bounded mapping of type (M) from X to X^* . Suppose that the mapping T satisfies the following condition:
There exists $R > 0$ such that

$$(3) \qquad (Tx, x) > 0 \text{ for } \|u\| > R .$$

Then the range of T is all of X^* .

Theorem 1 (Browder and Gupta). Let X be a Banach space, X^* its dual, T a bounded linear mapping of X into X^* which is monotone and angle-bounded. Then there exists a Hilbert space H , a continuous linear mapping S of X into H with S^* injective and a bounded skew-symmetric linear mapping B of H into H such that $T = S^*(I+B)S$ and the following inequalities hold:

(i) $\|B\| \leq a$, with a the constant of angle-boundednes of T

(ii) $\|S\|^2 \leq R$ if and only if for all u in X , $(Tu, u) \leq R\|u\|_X^2$

(iii) $[(I+B)^{-1}h, h]_H \geq (1 + a^2)^{-1} \|h\|_H^2$ for all h in H .

We are now in a position to state and prove our existence theorem.

Theorem 2. Let X be a Banach space and X^* its dual. Let $\{K_1, \dots, K_m\}$ be a finite family of bounded, linear, monotone and compact operators from X to X^* with constant of angle-boundedness $a \geq 0$ and $\|K_i\| \leq K_0$ for each i . Let $\{F_1, \dots, F_m\}$ be a corresponding finite family of continuous, bounded nonlinear operators from X^* to X which satisfy the following condition:
For every m -tuple $\{u_1, u_2, \dots, u_m\}$

(4) $$\sum_{i=1}^{m} (F_i(u), u_i) \geq - c \sum_{i=1}^{m} \|u_i\|^2_{X^*} + \sum_{i=1}^{m} (F_i(0), u_i)$$

where $u = \sum_{i=1}^{m} u_i$ and $c < (1 + a^2)^{-1} K_0^{-1}$.

Then the equation

(5) $$u + \sum_{i=1}^{m} K_i F_i u = 0$$

has a solution in X^* .

Proof: We first prove the following lemma.

Lemma 1. Let T be a continuous mapping from X to X^* such that $T = T_1 + T_2$ where T_1 satisfies the condition

(6) $$(T_1 x - T_1 y, x - y) \geq \phi(\|x - y\|) \quad \text{for all } x, y$$

$$\phi(r) \geq 0, \quad \phi(r) = 0 \quad \text{iff } r = 0$$

and T_2 is compact.

Then T is of type (M) .

<u>Proof</u>: Since T is continuous, it suffices to show that T satisfies condition (M_1) of Definition 1. Let $u_n \to u$ and $Tu_n \to w$ and $\limsup (Tu_n, u_n) \leq (w, u)$. Then we have

$$c(\|u_n - u\|) \leq (T_1 u_n - T_1 u, u_n - u)$$

$$= (Tu_n - Tu, u_n - u) - (T_2 u_n - T_2 u, u_n - u)$$

$$= (Tu_n, u_n) - (Tu_n, u) - (Tu, u_n - u) - (T_2 u_n - T_2 u, u_n - u) .$$

Since $u_n \rightharpoonup u$ and T_2 is compact, there exists a subsequence (which in turn will be denoted by u_n) such that $T_2 u_n \to y$. So we have

$$\begin{aligned} \limsup c(\|u_n - u\|) &\leq \limsup (Tu_n, u_n) - (w, u) \\ &\leq (w, u) - (w, u) \\ &\leq 0 \end{aligned}$$

which implies that $u_n \to u$. Since T is continuous $Tu_n \to Tu = w$, i.e. T satisfies condition (M_1) of Definition 1.

We now proceed to prove the main theorem. Since each K_i is angle-bounded, by Theorem 2 for each i there exists a Hilbert space H_i ,a continuous linear mapping $S_i : X \to H_i$ with S_i^* injective and a bounded linear skew-symmetric mapping B_i of H_i to H_i such that

(7) $$K_i = S_i^*(I + B_i)S_i, \quad \|B_i\| \leq a, \quad \|S_i\|^2 \leq K_0$$

and $[(I + B_i)^{-1} h_i, h_i]_{H_i} \geq (1 + a^2)^{-1} \|h_i\|_{H_i}$ for all h_i in H_i .

We form a Hilbert space H as the orthogonal direct sum $H = \sum_{i=1}^{m} \oplus H_i$. An element h of H is an m-tuple $\{h_1, \dots, h_m\}$ with h_i in H_i, while $\|h\|_H^2 = \sum_{j=1}^{m} \|h_j\|_{H_j}^2$.
We define a mapping $S : X \longrightarrow H$ by

$$Su = \{S_1 u,\ S_2 u, \dots,\ S_m u\} .$$

Then $S^* h = \sum_{i=1}^{m} S_i^* h_i$, $h = \{h_1, \dots, h_m\}$.

If u is a solution of (5), then (7) gives

(8) $$u + \sum_{i=1}^{m} S_i^* (I + B_i) S_i F_i u = 0 .$$

Since S^* is injective, there exists a unique h in H such that

(9) $$S^* h + \sum_{i=1}^{m} S_i^* (I + B_i) F_i S^* h = 0$$

which implies that

(10) $$h + \sum_{i=1}^{m} (I + B_i) S_i F_i S^* h = 0 .$$

Taking projections we get

(11) $$h_i + (I + B_i) S_i F_i S^* h = 0, \quad i = 1, 2, \dots, m$$

(12) $$(I + B_i)^{-1} h_i + S_i F_i S^* h = 0, \quad i = 1, 2, \dots, m .$$

This can be written as an operator equation

$$Th \equiv T_1 h + T_2 h = 0 \quad \text{in } H ,$$

where

$$(T_1 h)_i = (I + B_i)^{-1} h_i$$

$$(T_2 h)_i = S_i F_i S^* h .$$

(7) gives

$$[T_1 h, h]_H = \sum_{i=1}^{n} [(I+B_i)^{-1} h_i, h_i]_{H_i}$$

$$\geq (1+a^2)^{-1} \sum_{i=1}^{n} \| h_i \|^2$$

$$= (1+a^2)^{-1} \| h \|_H^2 ,$$

i.e.

(13) $$[T_1 h, h]_H \geq (1+a^2)^{-1} \| h \|_H^2 .$$

Also using (4) and (7) we get

$$[Th, h] = [T_1 h, h] + [T_2 h, h]$$

$$= \sum_{i=1}^{n} [(I+B_i)^{-1} h_i, h_i]_{H_i} + \sum_{i=1}^{N} [S_i F_i S^* h, h_i] H_i$$

$$\geq (1+a^2)^{-1} \| h \|_H^2 + \sum_{i=1}^{n} (F_i(S^* h), S_i^* h_i)$$

$$\geq (1+a^2)^{-1} \| h \|_H^2 - c \sum_{i=1}^{n} \| S_i^* h_i \|^2 + \sum_{i=1}^{n} (F_i(0), S_i^* h_i)$$

$$\geq (1+a^2)^{-1} \| h \|_H^2 - cK_0 \sum_{i=1}^{n} \| h_i \|^2 - \sum_{i=1}^{n} \| F_i(0) \| \, \| S_i^* h_i \|$$

$$\geq (1+a^2)^{-1} \| h \|_H^2 - cK_0 \| h \|_H^2 - \Big(\sum_{i=1}^{n} \| F_i(0) \|^2\Big)^{1/2} \Big(\sum_{i=1}^{n} \| S_i^* h_i \|^2\Big)^{1/2}$$

$$\geq [(1+a^2)^{-1} - cK_0] \| h \|_H^2 - \Big(\sum_{i=1}^{n} \| F_i(0) \|^2\Big)^{1/2} K_0^{1/2} \Big(\sum_{i=1}^{n} \| h_i \|_H^2\Big)^{1/2}$$

$$= [(1+a^2)^{-1} - cK_0] \| h \|_H^2 - \Big(\sum_{i=1}^{n} \| F_i(0) \|^2\Big)^{1/2} K_0^{1/2} \| h \|_H$$

$$= \left[c_0 - \left(\sum_{i=1}^{n} \frac{\| F_i(0) \|^2}{\| h \|_H^2} \right)^{1/2} K_0^{1/2} \right] \| h \|_H^2$$

where $c_0 = (1+a^2)^{-1} - cK_0 > 0$ by assumption on the constants. Hence there exists $R > 0$ such that $[Th, h] > 0$ for all $\|h\|_H > R$.

Since each K_i is compact, by Amann [1] each S_i in the splitting (7) is compact and therefore T_2 is compact. Thus the continuous operator T is the sum of the operator T_1 and T_2 where T_1 is linear and satisfies (6) and T_2 is compact. Therefore by Lemma 1 T is of typ (M) . Furthermore T is bounded because each S_i and F_i is bounded and satisfies the condition that $[Th, h] > 0$ for $\|h\|_H > R > 0$. So it follows by Proposition 1 that there exists a solution h in H of (10). This implies that S^*h is a solution of (8) and therefore of (5). This completes the proof.

<u>Remark</u>. Our Lemma 1 is similar to the Proposition 1.1 of [6] with the exception that our hypotheses are different.

R e f e r e n c e s

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