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"DISORDER" IN LATTICES OF BINUMERATIONS

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Abstract: The main result of this paper relates to results of Hájková and Palúch, regarding the ordering of consistency statements. We show that many of the same order properties are possessed by binumerations themselves, and by proof predicates. A result on independent sets of sentences in the Lindenbaum Sentence Algebra follows.

Key words: Arithmetization, binumeration, consistency, lattice, provability.

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Introduction. The technique of arithmetization of a theory T, which is central to many of the results of proof theory, depends for its success in any instance on the particular representations one chooses for the metamathematical functions and predicates. For Gödel's original representation of the proof predicate the "second incompleteness theorem" can be established for first order number theory, P. For Rosser's representation, extensionally identical, it cannot. Feferman [F 60] has established that for the set of results related to the second incompleteness theorem, the crutial decision is taken when membership in the set of

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axioms is represented. The Gödel results are obtained when the logical complexity of the axiom description is not too great. Terminology in this paper agrees generally with that in [F 60] and [H 71].

Feferman found that even among the "intensionally correct" representations of the axiom set; those for which the Gödel second theorem holds, there are still major differences which can be recognized by the theory. We use Feferman's term, <u>binumeration</u>, for a formula of T with one free variable, AO, for which it holds that:

If n is (the Gödel number of) an axiom of T, then $\vdash_T AO^{(n)}$.

If n is not (the Gödel number of) an axiom of T, then $\vdash_T -AO^{(n)}$.

We consider theories $T = \langle A, L \rangle$ where L is a first-order language and A is a <u>fixed</u> set of axioms binumerated by a formula obtained when a primitive recursive (p.r.) characteristic function is equated to 0, as $\alpha x = 0$. We are interested only in theories strong enough to represent p.r. functions and in intensional binumerations.

Hájková [H 71] studies the set of p.r. binumerations for first order arithmetic, P, preordered by the relation

$$\alpha \trianglelefteq_3 \beta \Longleftrightarrow \vdash CON_\beta \rightarrow CON_\alpha \qquad (PO_3)$$

where CON_α denotes the formula of P which naturally expresses consistency of P relative to the binumeration. Among other things, she shows that the equivalence classes of binumerations for the fixed axiom set A under the associated partial order form a lattice, Bin_P^A, and that the countable, atomless Boolean Algebra; hence any countable partially ordered set (poset) can be ordered isomorphically injected into Bin_P^A . This last result might be interpreted to mean that restricting to p.r. binumerations, although adequate to secure the proof of Gödel's theorem, leaves an unfortunate disorder among representations of the axiom set A , even in the case of recognizing the equivalence of the consistency problem within the theory. One can then ask whether the situation would change if one found conditions to restrict binumerations so that the equivalence of consistency could be established for all binumerations. Two other preorders of the binumerations spring to mind:

(PO_2) $\quad \alpha \trianglelefteq_2 \beta \Longleftrightarrow \vdash Prov_\alpha x \rightarrow Prov_\beta x$

(PO_1) $\quad \alpha \trianglelefteq_1 \beta \Longleftrightarrow \vdash \alpha x = 0 \rightarrow \beta x = 0$.

As in [F 60] and [H 71], we assume that such formulas as appear behind the turnstile in (PO_1) - (PO_3) lie in a particular p.r. extension of P , which we denote by P^+, but omit the dot used in [F 60]. Since we often have to use care in stating where formulas, and proofs, lie, we will use the notation $[E]^{P^+}$ of [F 60] for the result of applying the standard p.r. procedure for eliminating p.r. definitions from the formula E , of P^+ , to get a formula of P . Upper corners, as in $\ulcorner E \urcorner$, denote the numeral for the Gödel number of $[E]^{P^+}$, when E is a sentence of P^+ .

Several of our results depend upon an appplication of a form of the recursion theorem for primitive recursive functions, and upon its assertion being provable in P^+, as in Theorem 5.1 of [F 60] or, more appropriately, 2.12 of [F 62]. Since the recursion theorem is a statement regarding <u>indices</u> for functions, we must have available a p.r. indexing for the p.r. functions. We thus include in P^+, a symbol $\{m\}$ composed of braces and a natural number, to name the function, α, of that index, m. This is the only proper name for α in P^+, but we persist in writing α when there is no harm. This usage is at variance with [F 62] where $\{m\}$ is strictly a metamathematical object, but the same theorem as 2.12 holds in our P^+ and we cite it when required. In fact, the proof of Feferman's 2.12 for P^+ is a literal translation of the proof of the informal theorem in [Kl 58].

The p.r. binumerations under $\trianglelefteq_1$ clearly form a lattice under the obvious p.r. definitions of $\vee$ and $\wedge$, which give

$$(\alpha \wedge \beta)x = 0 \iff \alpha x + \beta x = 0$$
$$(\alpha \vee \beta)x = 0 \iff (\alpha x).(\beta x) = 0 \quad .$$

As remarked above, Hájková has shown that the structure under $\trianglelefteq_3$ is also a lattice. But we restrict the use of the symbols $\wedge$ and $\vee$ to the above meaning throughout the paper when they are applied to binumeration pairs. The same *) can be shown for $\trianglelefteq_2$ as follows. Given two p.r. binumerations α, β defined $\trianglelefteq_2$ -equivalent p.r. binumerations α', β',

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*) I.e. that the binumerations form a lattice under $\trianglelefteq_2$.

$$\alpha' x = \begin{cases} 0 \text{ if } (\alpha x = 0 \wedge \forall y_{\leq x} (\alpha y = \beta y)) \vee \exists y_1, y_2, y_{3 \leq x} (\alpha y_1 \neq \beta y_1 \\ \qquad \wedge \mathit{Prf}_{\alpha}(y_2, y_1) \wedge \mathit{Cl}(x, y_3)) \\ 1 \text{ otherwise} \end{cases}$$

where: $\mathit{Cl}(x, y_3)$ (is the formula of P^+ which) "says" that x is (the Gödel number for) some universal closure of the formula (whose Gödel number is) y_3, and $\mathit{Prf}_{\alpha}(x, y)$ (is the formula of P^+ which) says that x is a (Gödel number for a) proof of (the formula whose Gödel number is) y, from axiom binumeration α.

$\mathit{Prov}_{\alpha}(y)$ is $\exists x\, \mathit{Prf}_{\alpha}(x, y)$. Clearly, α and α' binumerate the same axiom set and the equivalence of their theorem sets can be proved in P^+, establishing $\alpha \trianglelefteq_2 \alpha'$ and $\alpha' \trianglelefteq_2 \alpha$. The same is done for β, β'. It is then easily shown that the *inf* of α, β in the $\trianglelefteq_2$ order is $\alpha' \wedge \beta'$, while the *sup* is $\alpha' \vee \beta'$, as in [H 71], theorems 2.19 and 2.21.

In Section I we shall give a new proof that any countable poset P, can be injected into the $\trianglelefteq_3$ lattice, and prove also that P can be injected into the $\trianglelefteq_2$ order of each $\trianglelefteq_3$ equivalence class, and again, into the $\trianglelefteq_1$-order of many of the $\trianglelefteq_2$ equivalence classes. These injections are carried out under an additional condition on the images which allows yet another proof of a theorem recently published in [M 72], and earlier in [K 62]. Our results may be taken to indicate there is little point in seeking conditions on p.r. binumerations solely to ensure the interdeducibility of of consistency statements , or even provability, since the same "disorder" will manifest itself at a more fundamental level. We are indebted to M. Hájková for several helpful comments and criticisms of the present work and particularly for greatly simplifying the original proof of Lemma 5. She has further pointed out that the imbedding results of the present paper extend to the lattices of all binumerations, of the fixed theory P , regardless of axiom set, which se denotes $\mathcal{Bin}_P^\infty$, in either of the two orders $\preceq_2$ or $\preceq_3$. This is true because, given a $\alpha \preceq_{2,3} \beta$ in $\mathcal{Bin}_P^\infty$, there is always β' binumerating the same axiom set as α so that $\alpha \preceq_{2,3} \beta' \preceq_{2,3} \beta$. The existence of β' can be proved by a recursion theorem construction like that used to prove Theorem 1. The results do not extend to $\mathcal{Bin}_P^\infty$ under $\preceq_1$ since there are pairs $\alpha \preceq_1 \beta$ in $\mathcal{Bin}_P^\infty$ in which β enumerates an axiom set obtained from the set A enumerated by α by the addition of some single non-axiom theorem.

In recent work, Jeroslow [J 73] investigates the suggestion of Kreisel [K 65] that the full description of a formal system should include the detailed rules for production of terms, formulas, and axioms within a context like a Post Normal System. From this viewpoint, the separate binumerations used in [F 60], [H 71], and the present paper would not correspond to the same formal system but, of course, there will be many different formal systems for the present system P , and they will be recognizably distinct in P . In another paper we will investigate the effect of restriction to sets of axioms obtainable by substitution into a finite set of schemes.

1. Injection theorems. We wish to show that arbitrary countable posets, P, can be order-isomorphically injected into the set of p.r. binumerations for arithmetic, ordered in any one of the three ways above. Theorem 1 covers all cases, by assuming an unspecified preorder relation (reflexive, transitive), $\preceq$, among the p.r. binumerations, subject to the conditions (0) and (1) below, and we consider P to be enumerated as $\{p_1, p_2, \dots, p_n, \dots\}$ with the inessential condition that P has a greatest element, p_1, and a least element, p_2. The construction of the injection proceeds by induction on a strengthened induction assumption necessary to provide that "sufficient space" is left at each stage to continue. To make this precise, we require a definition.

Definition 1: Let L be a lattice with order $\preceq$ and F_1, F_2 two finite subsets of L. Write

$$F_1 \preceq_e F_2 \qquad \text{for } (\exists x, y)(x \in F_1 \wedge y \in F_2 \wedge x \leq y)$$

$$F_1 \preceq_a F_2 \qquad \text{for } (\forall x, y)(x \in F_1 \wedge y \in F_2 \Longrightarrow x \leq y) .$$

Similar definitions hold for $\prec_e$ and $\prec_a$. A subset, M, of L, is called dispersed (in L) iff:

For each pair, F_1, F_2 of finite subsets of M,

whenever $\bigwedge_x F_1 \preceq \bigvee_y F_2$, then $F_1 \preceq_e F_2$.

Note that the word "disjoint" may be inserted before "subsets" and an equivalent definition results.

Each of the three preorder relations (PO_1) - (PO_2) is induced on the set of p.r. binumerations by a (p.r.) mapping, written $M_\alpha x$, from binumerations α to formulas $M_\alpha x$, of P^+, with at most one free variable, by the condition (0), below:

$$(0) \qquad \alpha_1 \trianglelefteq \alpha_2 \Longleftrightarrow \vdash_+ M_{\alpha_1} x \longrightarrow M_{\alpha_2} x \quad {}^{*)}$$

where the mapping M satisfies an additional condition, (1):

$$(1) \text{ If } \vdash_+ \alpha_1 x = 0 \longrightarrow \alpha_2 x = 0 , \quad \text{then } \vdash_+ M_{\alpha_1} x \longrightarrow M_{\alpha_2} x .$$

<u>Theorem 1</u>: Let P be an arbitrary countable poset, as above and M a mapping satisfying (1). Let $[\sigma_0, \alpha_0]$, $\sigma_0 \prec \alpha_0$; be a non-degenerate interval of the p.r. binumerations lattice under $\trianglelefteq$. Then P can be order isomorphically (and dispersively) injected into $[\sigma_0, \alpha_0]$.

<u>Proof</u>: We begin the construction of an injection σ , as follows:

$$\sigma(p_1) = \alpha_0$$
$$\sigma(p_2) = \sigma_0 \quad .$$

The set of images $\{\alpha_0, \sigma_0\}$ is a dispersed subset of the lattice of p.r. binumerations under $\trianglelefteq$. Suppose that σ has been defined for $p_1, p_2, \ldots, p_n$ and that $\sigma(\{p_1, p_2, \ldots, p_n\})$ is a dispersed subset. We must extend σ to p_{n+1} , and for this definition only, denote the strict order of P by $<$. Let

$A = \{\alpha_{i_1}, \ldots, \alpha_{I_1}\}$ be the images $\sigma(p_\nu)$ for $p_\nu > p_{n+1}$

$B = \{\beta_{j_1}, \ldots, \beta_{j_m}\}$ be the images $\sigma(p_\nu)$ for $p_\nu \parallel p_{n+1}$

$D = \{\sigma_{k_1}, \ldots, \sigma_{k_q}\}$ be the images $\sigma(p_\nu)$ for $p_\nu < p_{n+1}$

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*) Free variable formulas are intended to have the generality interpretation.

$\nu = 1, 2, \ldots, n$.

Define p.r. characteristic functions α, σ by

$$\alpha = \alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_l}$$

$$\sigma = \sigma_{k_1} \vee \sigma_{k_2} \vee \ldots \vee \sigma_{k_q} \quad .$$

The primitive recursive recursion theorem, Kleene [K 58], is used to define γx , to be designated $\sigma(p_{n+1})$ in such a way that if $\sigma(\{p_1, \ldots, p_n\})$ is dispersed, then so is $\sigma(\{p_1, \ldots, p_n, p_{n+1}\})$.

$$(2) \qquad \gamma x = \begin{cases} \sigma x & \text{until a clause below applied} \\ \sigma x & \text{if } \exists y_{\le x} [Prf(y, \ulcorner ND_\gamma (A,B,D,\gamma)\urcorner) \\ & \quad \wedge \forall z_{\le y}\, Prf(z, \ulcorner {}_\gamma ND(\gamma, A, B, D)\urcorner)] \\ \alpha x & \text{if } \exists y_{\le x} [Prf(y, \ulcorner {}_\gamma ND(\gamma, A, B, D)\urcorner) \\ & \quad \wedge \forall z_{\le y}\, Prf(z, \ulcorner ND_\gamma (A,B,D,\gamma)\urcorner)] \; . \end{cases}$$

Definition (2) should raise a number of questions, to be answered below. The formulas ${}_\gamma ND$ and ND_γ are p.r. expressions which correspond to the two ways in which dispersal could be lost when γ is added. Note that γx is, and remains, σx if the failure of dispersal is first attested by a proof, in P , of ND_γ or dispersal does not fail; and becomes and remains αx if dispersal fails and is first attested by a proof of ${}_\gamma ND$. The latter never happens, and the definition is (externally) fradulent, to the extent that it never does more than define a new name for the characteristic function of the same fixed axiom set.

Since, by the induction assumption, $\sigma(\{p_1, \ldots, p_n\})$ is dispersively injected, (0) and (1) give us:

(3) For each k, i, $\vdash_+ M_{\sigma_k} x \rightarrow M_{\alpha_i} x$

and hence, $\vdash_+ M_{\sigma} x \rightarrow M_{\alpha} x$.

Now, once we verify that (2) is a proper application of the p.r. recursion theorem, Theorem 2.12 of [F 62] will ensure that (2) is provable in P^+ and, hence,

(4) For each k, $\vdash_+ M_{\sigma_k} x \rightarrow M_{\gamma} x$

For each i, $\vdash_+ M_{\gamma} x \rightarrow M_{\alpha_i} x$

$ND_{\gamma}(A, B, D, \gamma)$ expresses that the set $A \cup B \cup D \cup \{\gamma\}$ is <u>not</u> dispersed because the definition fails for some pair F_1, F_2 of subsets with $\gamma \in F_2$. It is not difficult to see that, in the present case, this is adequately expressed by the finite disjunction of P^+ sentences of the form:

(5) $\forall x\,(M_{\alpha \wedge \bigwedge B^1} x \rightarrow M_{\sigma \vee \bigvee B^2 \vee \gamma} x)$

one for each pair B_1, B_2 of subsets of B with $B' \not\models_e B^2$
By convention, the upper corners translate this sentence into the language of P , and calculate its Gödel number. Similarly, $_{\gamma}ND(\gamma, A, B, D)$ expresses the other manner of dispersal failure, with $\gamma \in F_1$ by the finite disjunction of sentences

(6) $$\forall x\,(M_{\alpha \wedge \bigwedge B^1 \wedge \gamma}\, x \longrightarrow M_{\sigma \vee \bigvee B^2}\, x) \ .$$

For, if dispersal does fail when γ is added there will be an inequality (with γ on the left or right, but taken left for illustration) so that for disjoint pairs of sets $A^1, A^2 \subseteq A$; $B^1, B^2 \subseteq B$; $D^1, D^2 \subseteq D$;

$$\gamma \wedge \bigwedge A^1 \wedge \bigwedge B^1 \wedge \bigwedge D^1 \preceq \bigvee A^2 \vee \bigvee B^2 \vee \bigvee D^2 \ .$$

Since this expresses failure of dispersion and, by (4) $\gamma \preceq \alpha_i$, $i = 1, \ldots, \ell$, we see that $A^2 = \phi$. Again by (4), $\sigma_k \preceq \gamma$, $k = 1, \ldots, q$, so that unless $D^1 = \phi$ the induction assumption is violated. Thus, a fortiori, an inequality of the form

$$\gamma \wedge \alpha \wedge \bigwedge B^1 \preceq \sigma \vee \bigvee B^2$$

must follow, where $B^1 \not\equiv_e B^2$, and this is of the form (6). Finally, note that the incorporation of these formulas into the definition of γ requires only that the <u>index</u> of γ appear in the defining clauses and, hence (2) is a proper application of the p.r. recursion theorem.

We let $\sigma(r_{n+1})$ be γ and first show that the set $\sigma(\{r_1, \ldots, r_n, r_{n+1}\})$ has the dispersion property, and next that it is order isomorphically injected. If dispersion fails, there will be two disjoint sets, B^1 and B^2 , as above "expressing" the failure of dispersion. There are two cases to consider.

(i) $\gamma \in F_1$. Since dispersion fails, we have

(7) $$\vdash_{+} M_{\alpha \wedge \bigwedge B^1 \wedge \gamma}\, x \longrightarrow M_{\sigma \vee \bigvee B^2}\, x$$

and hence

$$\vdash_{+} {}_{\gamma}ND(\gamma, A, B, D) \ .$$

Thus, there is a natural number, m , for which

$$\vdash_{+} Prf(0^{(m)}, \ulcorner {}_{\gamma}ND(\gamma, A, B, D)\urcorner) \ .$$

and, by provability in P^{+} of (2),

$$\vdash_{+} \forall x\,(m < x \longrightarrow \gamma x = \alpha x) \ .$$

However, since γ and α are, in fact, equal for all arguments, we get in the usual way,

(8) $$\vdash_{+} \forall x\,(\gamma x = \alpha x) \ .$$

Hence,

(9) $$\vdash_{+} \forall x\,(\wedge B^{1} \wedge \alpha \wedge \gamma\,(x) = \wedge B^{1} \wedge \alpha\,(x))$$

and, by (1), (7), (9),

(10) $$\vdash_{+} M_{\wedge B^{1} \wedge \alpha}\, x \longrightarrow M_{\vee B^{2} \vee \delta}\, x$$

which contradicts the assumption of dispersion of $\sigma(\{n_1, n_2, \dots, n_m\})$.

(ii) $\gamma \in F_2$. In this case we have, similarly,

(11) $$\vdash_{+} M_{\wedge B^{1} \wedge \alpha}\, x \longrightarrow M_{\vee B^{2} \vee \delta \vee \gamma}\, x$$

and, by the same sort of argument, obtain

(12) $$\vdash_{+} \forall x\,(\delta x = \gamma x)$$

$$\vdash_{+} \forall x\,(\vee B^{2} \vee \delta \vee \gamma\,(x) = \vee B^{2} \vee \delta\,(x))$$

and, again by (1), (11), (12),

(13) $$\vdash_{+} M_{\wedge B^1 \wedge \alpha}\, x \longrightarrow M_{\vee B^2 \vee \delta}\, x$$

contradicting the induction assumption. This proves the dispersion of $\sigma(\{p_1, \ldots, p_m, p_{m+1}\})$.

It is immediate that the correct (strict) order properties are possessed by the new element σ , by (4), the contradiction yielded by (8) and (12), and the following. Suppose that γ were comparable to some β_j ; then either (a): $\gamma \preceq \beta_j$ or (b): $\beta_j \preceq \gamma$. In case (a) we have

(14) $$\vdash_{+} M_{\gamma}\, x \longrightarrow M_{\beta_j}\, x$$

which leads again to the contradiction of (7) - (10). In case (b) we have

(15) $$\vdash_{+} M_{\beta_j}\, x \longrightarrow M_{\gamma}\, x$$

leading to the contradiction of (11) - (13). This shows that the injection σ can be extended and thus, by induction, the proof of Theorem 1 is complete.

Definition 2: A subset, M , of a lattice L will be called independent (absolutely) independent in [M 72]) if, for no two disjoint subsets F_1, F_2 of M , may we have $\bigwedge_{F_1} \preceq \bigvee_{F_2}$.

Corollary 2: If the poset P , of Theorem 1, is an antichain (no two elements comparable), then $\sigma(P)$ is an independent set, in the sense of Definition 2.

By Theorem 3, below, it will be seen that it suffices to prove a restricted version of Theorem 1, in the case when P is an antichain, but the proof is not essentially simpler.

Theorem 3 was formulated and proved by M. Adams, of Bristol University, in response to a question asked by Myhill. The same result follows from Lemma 4.1 of [B 67] and Theorem 3.10 of [BH 67]. However, we give Adams' proof here to make the result available in the present context.

Theorem 3: If P is an arbitrary countable poset, and L is the free distributive lattice on a countable independent set of generators, then P can be order isomorphically injected into L.

Proof: The proof is like that of Theorem 1, and we use a similar notation. The induction assumption is, again that the subposet $\{p_1,\dots,p_n\}$ has been order isomorphically injected, by σ, onto a dispersed subset of L. Let

$$A = \{p_i \mid 1 \le i \le n \wedge p_{n+1} \prec p_i\}, \quad A' = \sigma(A)$$

$$D = \{p_k \mid 1 \le k \le n \wedge p_k \prec p_{n+1}\}, \quad D' = \sigma(D)$$

$\alpha = \bigwedge\sigma(A) = \bigwedge_{A'}$ (instead of the previous notation $\bigwedge A'$)

$\delta = \bigvee\sigma(D) = \bigvee_{D'}$.

We use a lemma of Balbes [B 67], Lemma 4.5.

Lemma: In the free distributive lattice on an independent set of generators, if we have

$$\bigwedge_{S_1} \vee \bigwedge_{S_2} \vee \dots \vee \bigwedge_{S_k} \preceq \bigwedge_{T_1} \vee \bigwedge_{T_2} \vee \dots \vee \bigwedge_{T_h} \tag{17}$$

where the sets $S_1,\dots,S_k$; $T_1,\dots,T_h$ are finite, non-empty, sets of generators, then for each S_i , there is a T_j so that T_j is a subset of S_i .

To prove the Theorem, note that by the induction assumption of dispersion, $\delta < \alpha$. Choose a "new" generator, q , not yet used in forming $\sigma(p_i)$, $1 \le i \le m$, and construct

$$\sigma(p_{m+1}) = \gamma = \alpha \wedge (\delta \vee q) = \delta \vee (\alpha \wedge q) . \tag{18}$$

From the independence of the generators, $\delta \prec \gamma \prec \alpha$. Suppose next, that for some $p_j \| p_{m+1}$, $\sigma(p_j)$ is related to γ . Suppose $\sigma(p_j) \preceq \gamma$, i.e.

$$\sigma(p_j) \preceq \alpha \wedge (\delta \vee q) \preceq \delta \vee q . \tag{19}$$

Then, for the appropriate finite sets of generators,

$$\bigwedge_{S_1} \vee \ldots \vee \bigwedge_{S_R} \preceq \bigwedge_{T_1} \vee \ldots \vee \bigwedge_{T_S} \vee q . \tag{20}$$

By Balbes´Lemma, for each S_i , there is a T_j (or $\{q\}$), $T_j \subseteq S_i$ (or $\{q\} \subseteq S_i$). However, the parenthetical remark is not possible and, again by the lemma, we conclude $\sigma(p_j) \preceq \delta$. This is not possible, since the induction assumption of dispersion would require $\sigma(p_j) \preceq \sigma(p_k)$ for some $p_k \in D$ and by the induction assumption on order, $p_j \preceq p_k$. Since we must have $p_k \preceq p_{m+1}$,we get a contradiction to the choice of $p_j \| p_{m+1}$.

If we suppose $\gamma \preceq \sigma(p_j)$,then

$$\alpha \wedge q \preceq (\alpha \wedge q) \vee \delta \preceq \sigma(p_j)$$

and, for suitably chosen generator sets,

$$\bigwedge_{T'_1} \vee \ldots \vee \bigwedge_{T'_s} = (\bigwedge_{T_1} \vee \ldots \vee \bigwedge_{T_s}) \wedge q \preceq \bigwedge_{S_1} \vee \ldots \vee \bigwedge_{S_k} \tag{21}$$

where $T'_j = T_j \cup \{q\}$, $1 \leq j \leq s$. In this case, Balbes' lemma requires for each T'_j, an S_i with $S_i \subseteq T'_j$. But since q is a new generator, $S_i \subseteq T_j$, and the lemma requires $\alpha \preceq \sigma(p_j)$, a contradiction is produced as before. Thus the set $\{p_1, \ldots, p_n, p_{n+1}\}$ is order isomorphically injected in L. It remains to show that the image set is dispersed.

It suffices to consider disjoint sets of $\sigma(\{p_1, \ldots, p_{n+1}\})$, $\bigwedge_{F_1} \preceq \bigvee_{F_2}$. There are three cases to consider, (i): $\gamma \notin F_1 \cup F_2$, which is trivial, (ii) $\gamma \in F_1$; and (iii): $\gamma \in F_2$. Suppose (ii). Write

(22) $$\bigwedge_{F_1-\{\gamma\}} \wedge [\sigma \vee (\alpha \wedge q)] \preceq \bigvee_{F_2} .$$

Hence

$$\bigwedge_{F_1-\{\gamma\}} \wedge (\alpha \wedge q) \preceq \bigvee_{F_2} .$$

Write

$$\bigwedge_{F_1-\{\gamma\}} \wedge \alpha = \bigwedge_{S_1} \vee \ldots \vee \bigwedge_{S_r} \text{ and } \bigvee_{F_2} = \bigwedge_{T_1} \vee \ldots \vee \bigwedge_{T_s}$$

where the sets $S_1, \ldots, S_r$; $T_1, \ldots, T_s$ are finite sets of generators, excluding q. Then

(23) $$\bigwedge_{S'_1} \vee \ldots \vee \bigwedge_{S'_r} \preceq \bigwedge_{T_1} \vee \ldots \vee \bigwedge_{T_s}$$

where $S'_i = S_i \cup \{q\}$. By Balbes' Lemma, for each S'_i, there is a T_j, $T_j \subseteq S'_i$. But, also, $T_j \subseteq S'_i - \{q\}$, and hence,

(24) $$\bigwedge_{F_1-\{\gamma\}} \wedge \bigwedge_{A'} \preceq \bigvee_{F_2}$$

Now, by the induction assumption of dispersion, there is a pair, $\langle \varepsilon, \phi \rangle$, $\varepsilon \in (F_1 - \{\gamma\}) \cup A'$, $\phi \in F_2$, so that $\varepsilon \preceq \phi$.

If $\varepsilon \in F_1$,the same pair $\langle \varepsilon, \phi \rangle$ is effective for the sets F_1, F_2 in the dispersion definition. If $\varepsilon \notin F_1$, then $\varepsilon \in A'$ and since $\gamma \preceq \varepsilon$ the pair $\langle \gamma, \phi \rangle$ is effective, since $\gamma \in F_1$.

If (iii), $\gamma \in F_1$, write

$$(25) \qquad \bigwedge_{F_1} \preceq \bigvee_{F_2 - \{\gamma\}} \vee \delta \vee (\alpha \wedge \varphi)$$

and, by the Balbes'Lemma argument, $\bigwedge_{F_1} \preceq \bigvee_{F_2-\{\gamma\}} \vee \delta = \bigvee_{F_2-\{\gamma\}} \vee \bigvee_{D'}$. By the induction assumption, there is a pair, $\varepsilon \in F_1$, $\phi \in (F_2 - \{\gamma\}) \cup D'$, so that $\varepsilon \leq \phi$. If $\phi \in F_2$ then the pair $\langle \varepsilon, \phi \rangle$ is effective and, if not $\phi \in D'$ and $\phi \preceq \gamma$, $\gamma \in F_2$, so that the pair $\langle \varepsilon, \gamma \rangle$ is effective. Thus Theorem 3 is proved.

2. <u>Separation theorems</u>

A requirement of the basic injection result, Theorem 1, is a non-degenerate interval of binumerations $[\delta_0, \alpha_0]$, $\delta_0 \prec \alpha_0$,into which the injection takes place. The existence of such intervals for the order $\preceq_3$, induced by consistency statements, is well known, and the properties of the lattices of binumerations under this order are explored in Hájková [H 71]. In this section we show the existence of non-degenerate intervals (within each $\preceq_3$ equivalence class) in the $\preceq_2$ order, and, again (within almost all $\preceq_2$ equivalence classes) in the $\preceq_1$ order.

We first show that it is possible to choose inequivalent binumerations, $\beta \prec_1 \beta'$, which are equivalent in the $\preceq_2$ sense, $\beta \equiv_2 \beta'$. This is quite easy for functions β

which satisfy a mild additional restriction; that for <u>some</u> formula, F, of P, which is not an axiom, we shall be able to prove, in P^+, that neither F nor any conjunction of F's is an <u>axiom</u>. I.E. we require:

$$(26) \qquad \vdash_+ \neg \exists y (\mathrm{CIF}\, y \wedge \beta(y) = 0)$$

where CIF is the obvious p.r. predicate. Although there will be many p.r. characteristic functions of the axiom set not satisfying (26) it is a very mild restriction which amounts to requiring the theory to recognize that its <u>axiom</u> set is contradiction free, with respect to propositional deduction, i.e. it is sort of pre-consistency obtained by enforcing an elementary degree of reasonableness on the manner of expressing the axiom set.

<u>Lemma 4:</u> If β is a p.r. characteristic function of the axiom set satisfying (26), there is another p.r. characteristic function, β', for the same axiom set, satisfying:

$$(27)\ (\mathrm{i}) \qquad \vdash_+ \forall x (\beta x = 0 \rightarrow \beta' x = 0)$$

$$(\mathrm{ii}) \qquad \nvdash_+ \forall x (\beta x = 0 \leftarrow \beta' x = 0)$$

$$(\mathrm{iii}) \qquad \vdash_+ \forall x (\mathrm{Prov}_\beta x \leftrightarrow \mathrm{Prov}_{\beta'} x) .$$

<u>Proof</u>: Let F be as required by (26). Then β' has a simple definition from β, by cases:

$$(28) \quad \beta'(x) = \begin{cases} 0 \text{ if } \mathrm{CIF}\, x \wedge \exists y_{\leq x} \mathrm{Prf}_\beta(\dot{y}, \ulcorner 0 = 1 \urcorner) \\ x \text{ otherwise} \end{cases} .$$

The definition by cases is provable is P^+ and hence, immediately,

(29 $$\vdash_+ \beta x = 0 \rightarrow \beta' x = 0$$

$$\vdash_+ Prov_\beta x \rightarrow Prov_{\beta'} x \quad .$$

To show the reverse implication in (iii) it is only necessary to observe the following free-variable deduction in P^+ :

$$Prov_{\beta'} x, \neg Prov_\beta x$$

$$\exists y \, Prf_\beta (y, \ulcorner 0 = 1 \urcorner)$$

$$\neg CON_\beta$$

$$Prov_\beta (x)$$

yields:

(30) $$\vdash_+ Prov_{\beta'} x \rightarrow Prov_\beta x \quad .$$

Finally, if $\vdash_+ \beta' x = 0 \rightarrow \beta x = 0$ then, using (28) we obtain the contradiction $\vdash_+ CON_\beta$.

Next, we turn to the production of inequivalent binumerations in the $\preceq_2$ order within any $\equiv_3$ equivalence class.

<u>Lemma 5:</u> Let β be any p.r. characteristic function of the axiom set. There is another, β' , for the same axiom set so that:

(31) (i) $$\vdash_+ \forall x (Prov \, x \rightarrow Prov \, , x)$$

(ii) $$\nvdash_+ \forall x (Prov \, x \leftarrow Prov \, , x)$$

(iii) $$\vdash_+ CON_\beta \leftrightarrow CON_{\beta'} \quad .$$

Proof: As in Definition 1.10 of [H 71], we take the Rosser Sentence for the binumeration β to be that "solution" R_β (a sentence of P^+) of the basic diagonalization lemma for which

$$(32) \quad \vdash_+ R_\beta \leftrightarrow (\forall y)[Prf_\beta(\ulcorner R_\beta \urcorner, y) \rightarrow (\exists z)(z < y\, Prf_\beta(\ulcorner \neg R_\beta \urcorner, z)] .$$

Then, as in Theorem 1.14 of [H 71] the basic properties of R_β can be summarized as:

$$(33) \quad \begin{aligned} &\vdash_+ CON_\beta \leftrightarrow \neg Prov_\beta \ulcorner R_\beta \urcorner \\ &\vdash_+ CON_\beta \leftrightarrow \neg Prov_\beta \ulcorner_\beta R \urcorner \end{aligned}$$

Now, we form the extension theory $P' = P \cup \{[CON_\beta]^{P^+}\}$ and let ε denote the natural binumeration of the theory P', with R_ε as associated Rosser Sentence. We form another binumeration β', of P, as follows

$$(34) \quad \beta' x = \begin{cases} 0 \text{ if } \beta x = 0 \text{ or } (\exists y_{\leq x})[Prf_\varepsilon(\ulcorner R_\varepsilon \urcorner, y) \\ \qquad \wedge (\forall z_{<y}) \neg Prf_\varepsilon(\ulcorner \neg R_\varepsilon \urcorner, z) \wedge Cl(\ulcorner R_\beta \urcorner, x)] \\ 1 \text{ otherwise} . \end{cases}$$

We obtain immediately from (34):

$$(35) \quad \vdash_+ Prov_\beta x \rightarrow Prov_{\beta'} x$$

and, hence,

$$(36) \quad \vdash_+ CON_{\beta'} \rightarrow CON_\beta \quad .$$

To reverse the implication in (36) we formalize in P^+ the following informal deduction. If $\neg CON_{\beta'}$, then from (34) $Prov_\beta \ulcorner \neg R_\beta \urcorner$ (since $P \cup \{[R_\beta]^{P^+}\}$ is inconsistent). From (33), this yields $\neg CON_\beta$, and hence we have:

(37) $\vdash_{+} CON_{\beta'} \leftrightarrow CON_{\beta}$.

Finally, if the reverse implication of (35), ie. $Prov_{\beta'} x \rightarrow Prov_{\beta} x$, were P^{+} provable, then:-

$$\vdash_{+} Prov_{\beta'} \ulcorner R_{\beta} \urcorner \rightarrow Prov_{\beta} \ulcorner R_{\beta} \urcorner \; .$$

By (33),

$$\vdash_{+} Prov_{\beta'} \ulcorner R_{\beta} \urcorner \rightarrow \neg CON_{\beta}$$

or

(38) $\vdash_{+} CON_{\beta} \rightarrow \neg Prov_{\beta'} \ulcorner R_{\beta} \urcorner$.

However, by (32) and (34)

(39) $\vdash_{+} \neg Prov_{\beta'} \ulcorner R_{\beta} \urcorner \rightarrow R_{\varepsilon}$

which, together with (38) yields the contradiction:

$$\vdash_{+} CON_{\beta} \rightarrow R_{\varepsilon} \; .$$

This completes the proof of Lemma 5.

3. Conclusion

We can state immediate consequence of the separation results in Section 2, and Theorem 1, in the following corollaries. The first, for the $\dashv_3$ ordering, was already obtained by Hájková [H 71]. To obtain ot from Theorem 1 requires a trivial change in Conditions (0) and (1), to reverse the implication on the right.

Corollary 6: Any countable poset, P , may be order-isomorphically injected into the lattice of binumerations in its $\dashv_3$ ordering.

Corollary 7: For any countable poset, P , and any equivalence class, E , of binumerations under the $\dashv_3$ ordering, P may be order-isomorphically injected into E in its $\dashv_2$ ordering.

Corollary 8: For any countable poset, P , and any equivalence class, F , of binumerations under the $\preceq_2$ ordering, where elements of F satisfy, in addition, the preconsistency condition of Lemma 4, P may be order-isomorphically injected into F , in its $\preceq_1$ order.

The order of the lattice of binumerations, $\preceq_3$ is just the anti-isomorphic image of the order among consistency statements for P in the Lindenbaum Sentence Algebra (LSA) for P . The orders $\preceq_2$ and $\preceq_1$ are similarly isomorphic copies of the order in the Lindenbaum Algebra of formulas with one free variable (LFA). As noted earlier, the dispersion condition on the injection of Theorem 1, when applied to a countable anti-chain P , yields a set of injection images which are independent in the parent algebra. There is a history to this problem. In a recent paper, Myhill [My 71] includes a proof of the existence of a set of Σ_1^0 sentences independent in the LSA of P . He comments that the result can be strengthened to prove the existence of a single Σ_1^0 formula, with one free variable, Ax , for which the set $\{A0, A0^{(1)}, \ldots, A0^{(n)}, \ldots\}$ is independent. Kripke has proved this latter result in [Krp 62] and credits earlier solutions to Mostowski [Mo 60] and to Feferman and Scott. Another proof is implicit in Lemma 3.1 and Theorem 3.1 of [J 72].

The uniformity condition leading to the set $\{A0, A0^{(1)}, \ldots, A0^{(n)}, \ldots\}$ can be built into the proof of our Theorem 1, by including an additional variable in the definition of γ, (2) .

This variable would be used to "regulate" the lengths of the clauses ${}_{\gamma}ND$ and ND_{γ} to provide for the successive injection of more elements of the anti-chain P . Hence we state one further corollary which "improves" the Kripke result to formulas with no unbounded quantifiers.

Corollary 9: There is a p.r. formula [F 60] of P with two free variables, Axy , so that the set $\{Ax0, Ax0^{(1)}, \ldots, Ax0^{(n)}, \ldots\}$ is independent in the LFA.

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