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REGULARITY AND EXTENSION OF MAPPINGS IN SEQUENTIAL SPACES

R. FRIČ, Žilina

Abstract: The class of all topological spaces Y with unique sequential limits that satisfy the following property ($*$) is characterized:

($*$) For each sequential space X and each continuous mapping f of a dense subspace X_0 of X into Y if f can be continuously extended onto each subspace $X_0 \cup (x)$, $x \in X$, then it can be continuously extended onto X.

Key words: Sequential space, regular soace, extension of mappings, convergence.

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§ 0

For the reader's convenience we shall in this section briefly outline how to construct a sequential space by means of convergent sequences (cf. [4], [5], [6]) and recall some facts about sequential spaces needed in the sequel.

In a non-empty set X we define a convergence $\mathcal{L}$, i.e. we declare some sequences of points to converge to their limit points such that:

($\mathcal{L}_1$) - constant sequences converge,

($\mathcal{L}_2$) - subsequences of convergent sequences converge,

($\mathcal{S}$) - the set of limit points of any sequence is sequentially closed.

Notice that the condition

($\mathcal{L}_0$) - each sequence has at most one limit point

implies ($\mathcal{S}$). The convergence of sequences in every topological space satisfies Conditions ($\mathcal{L}_1$),($\mathcal{L}_2$),($\mathcal{S}$), while it may not satisfy ($\mathcal{L}_0$).

Now, the set of all sequentially open sets forms a topology for X. In this sequential space a sequence $\langle x_n \rangle$ converges to a point x iff every subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ contains a subsequence $\langle x''_n \rangle$ which $\mathcal{L}$-converges to x. The convergence $\mathcal{L}$ is sometimes called a priori and the convergence in the sequential space X is called a posteriori. Similarly, as in [6, Lemma 5] it can be proved that if f_1, f_2 are continuous mappings of a sequential space X into a sequential space Y with unique sequential limits such that f_1 and f_2 coincide on a dense subset of X, then they are equal. Finally, to each topological space Y, a sequential space sY corresponds such that if f is a mapping of a sequential space X into Y, then f is continuous iff f is continuous as a mapping of X into sY. The topology of the sequential space sY consists of all sequentially open sets in Y.

§ 1

It is well-known that if f is a continuous mapping of a dense subspace X_0 of a topological space X into a regular space Y and f can be continuously extended onto each subspace $X_0 \cup (x)$, $x \in X$, then f can be continuously extended onto X. Moreover, the regularity assumption is essential (cf.[1, pp.857]). However, if X is supposed to be sequential, then the regularity condition can be weakened and, in view of § 0, the weaker condition should concern the space sY.

Definition 1. A sequential space Y in which a sequence $\langle y_n \rangle$ converges to y whenever every closed neighbourhood of y contains y_n for all but finitely many n is said to be convergence regular or briefly c-regular. A topological space Y is called c-regular if sY is c-regular.

Theorem 1. In a sequential space Y the following conditions are equivalent:

(a) Y is c-regular.

(b) If $\langle y_n \rangle$ is a sequence and $y \in Y - \overline{U(y_n)}$, then there is a closed neighbourhood O of y such that $y_n \in Y - O$ for infinitely many n.

(c) If $\langle y_n \rangle$ is a sequence and $y \in Y - \overline{U(y_n)}$, then there is a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that y and $U(y'_n)$ can be separated by disjoint open sets.

Proof. (a)$\Longrightarrow$(b). If $\langle y_n \rangle$ is a sequence and $y \in Y - \overline{U(y_n)}$, then $\langle y_n \rangle$ does not converge to y and (b) follows from Definition 1.

(b)$\Longrightarrow$(a). Let $\langle y_n \rangle$ be a sequence and let every closed neighbourhood of y contain y_n for all but finitely many n. Then the same holds for any subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ and by (b) we have $y \in \overline{U(y'_n)}$ for any such $\langle y'_n \rangle$. It follows that $\langle y_n \rangle$ converges to y.

The proof of (b)$\Longleftrightarrow$(c) is easy and omitted.

Theorem 2. Let X_0 be a dense subspace of a sequential space X. Let f be a continuous mapping of X_0 into a c-regular space Y. If f can be continuously extended onto each subspace $X_0 \cup (x)$, $x \in X$, then it can be continuously extended onto X.

Proof. Without loss of generality we can obviously suppose that $Y = \mathcal{s}Y$. In this proof the bar always denotes the closure in X, Y respectively. Let for each $x \in X$ there be a continuous extension f_x of f onto the subspace $X_0 \cup (x)$. From the continuity of f_x it follows that

(i) $f_x(x) \in \overline{f[A]}$ for each $A \subset X_0$, $x \in \overline{A}$.

Moreover,

(ii) if $y \in X_0 \cup (x)$ and $O \subset Y$ is an open set such that $f_x(y) \in O$, then $y \in \overline{f^{\leftarrow}[O]}$,

since $f^{\leftarrow}[O] = f_x^{\leftarrow}[O] \cap X_0$ and X_0 is dense in X. Denote by F the mapping defined on X as follows:

(iii) $F(x) = f(x)$ for $x \in X_0$,

$F(x) = f_x(x)$ for $x \in X - X_0$.

We shall prove that F is continuous (i.e. sequentially continuous). Let $x = \lim x_n$ in X and suppose that, on the contrary, $F(x) \in Y - \overline{\bigcup(F(x'_n))}$ for some subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$. Since Y is c-regular, there is a subsequence $\langle x''_n \rangle$ of $\langle x'_n \rangle$ and disjoint open sets $O_1, O_2 \subset Y$ such that $F(x) \in O_1$, $\bigcup(F(x''_n)) \subset O_2$. From (ii) it follows that for $n \in N$ we have $x''_n \in \overline{f^{\leftarrow}[O_2]}$ and hence

$x \in \overline{U(x''_m)}$ implies $x \in \overline{f^{\leftarrow}[O_2]}$. On the other hand, from (i) and (iii) follows $F(x) \in \overline{f[f^{\leftarrow}[O_2]]} \subset \overline{O_2} \subset Y - O_1$. This is a contradiction and the theorem is proved.

Theorem 3. Let Y be a topological space with unique sequential limits which is not c-regular. Then there exist a sequential space X , a dense subspace X_0 of X and a continuous mapping f of X_0 into Y such that f can be continuously extended onto each subspace $X_0 \cup (x)$, $x \in X$, but cannot be continuously extended onto X .

Proof. Again, it is sufficient to prove the theorem in the case of $Y = sY$. According to (b) of Theorem 1 there is a sequence $\langle y_n \rangle$ and a point y in Y such that $y \in Y - \overline{U(y_n)}$ and for every closed neighbourhood O of y we have $y_n \in O$ for all but finitely many n . There are three possibilities:

1. The sequence $\langle y_n \rangle$ is totally divergent. We can suppose without loss of generality that $\langle y_n \rangle$ is one-to-one. Then there is a natural m such that the set $Y - \bigcup_{n>m} (x_n)$ is dense in Y . For otherwise there is a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that the points y'_n are isolated. Thus $\bigcup (y'_n)$ is a closed-open set not containing y and we have a contradiction. Now, we enlarge the convergence in Y declaring all subsequences of $\langle y_n \rangle$ to be convergent to y and denote by X the induced sequential space. Clearly, the topology of X is coarser than that of Y . Finally, let $X_0 = Y - \bigcup_{n>m} (y_n)$, let f be the identical mapping on X_0 considered as a mapping of X_0 into Y . Since X_0 is dense in Y , it is also dense in X . The identi-

cal mapping on each subspace $X_0 \cup (y_n)$, $n > m$,is clearly the uniquely determined continuous extension of f , but it cannot be continuously extended onto X .

2. There is a one-to-one subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ converging to a point $q \in Y$. Since Y is a sequential space with unique sequential limits, the set (q) is closed and the open subspace $Y' = Y - (q)$ is sequential (cf. [2]). It is easy to see that in Y' we have $y \in Y' - \overline{\cup (y'_n)}$ and $y'_n \in O$ for all but finitely many n for every closed neighbourhood O of y . Now we proceed similarly as in 1 .

3. There is a point $z \in Y$ and a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $y'_n = z$, $n = 1, 2, \dots$. Notice that Y is not Hausdorff in this case, since y and z cannot be separated by disjoint open sets. Let X be the union of a one-to-one double sequence $\langle x_{mn} \rangle$ a one-to-one sequence $\langle x_m \rangle$ and a point x . We introduce into X a sequential topology by means of convergent sequences as follows: for each $a \in X$ the constant sequence $\langle a \rangle$ converges to a , for every m each subsequence $\langle a_n \rangle$ of $\langle x_{mn} \rangle$ converges to x_m , each subsequence of $\langle x_m \rangle$ converges to x . Denote $X_0 = X - \cup (x_m)$ the dense subspace of X and define a mapping f of X_0 into Y in the following way: $f(x_{mn}) = z$, $f(x) = y$. Then f can be uniquely continuously extended onto each subspace $X_0 \cup (x_m)$, but cannot be continuously extended onto X . This completes the proof.

<u>Corollary</u>. Let Y be a topological space with unique sequential limits. Then the following conditions are equivalent:

(d) Y is c-regular.

($*$) For each sequential space X and each continuous mapping f of a dense subspace X_0 of X into Y if f can be continuously extended onto each subspace $X_0 \cup (x)$, $x \in X$, then it can be continuously extended onto X.

Example. Let Y be the real line the topology of which is enlarged in such a way that the sets $(-\varepsilon, \varepsilon) - \cup (\pi/n)$, $\varepsilon > 0$, are also neighbourhoods of 0. Then Y is a Hausdorff (Fréchet) sequential space which is not c-regular.

§ 2

In this section we shall study further properties of c-regular spaces. We start with mutual relations between c-regularity and the separation axioms.

Theorem 4. A regular space is c-regular.

Proof. Let Y be a regular space. If $\langle y_n \rangle$ is a sequence and $y \in Y - \overline{\cup (y_n)}$ in sY, then there exists a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $y \in Y - \overline{\cup (y'_n)}$ in Y. Since Y is regular, y and $\cup (y'_n)$ can be separated by disjoint open sets in Y and hence in sY.

For our purpose we shall generalize the notion of sequential regularity introduced by J. Novák ([6]) for convergence closure spaces.

Definition 2. A topological space Y is said to be sequentially regular if the convergence of sequences in Y is projectively generated by the set of all continuous functions on Y, i.e. $\langle y_n \rangle$ converges to y in Y whenever for each continuous function f on Y we have $f(y) = \lim f(y_n)$.

Notice that a sequentially regular space with unique sequential limits is completely Hausdorff.

Theorem 5. A sequentially regular space is c-regular.

Proof. Let Y be a sequentially regular space. If $\langle y_n \rangle$ is a sequence and $y \in Y - \overline{\cup \langle y_n \rangle}$ in sY, then $\langle y_n \rangle$ does not converge to y in Y. Consequently, there is a continuous function f on Y such that $\langle f(y_n) \rangle$ does not converge to $f(y)$. Hence there is a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $f(y) \in R - \overline{\cup (f(y'_n))}$ and from the regularity of R follows the existence of disjoint open sets $O_1, O_2 \subset R$ such that $f(y) \in O_1$, $\cup (f(y'_n)) \subset O_2$. The sets $f^{\leftarrow}[O_1]$, $f^{\leftarrow}[O_2]$ are open in sY and separate y and $\cup (y'_n)$.

Proposition 1. A c-regular sequential Hausdorff space need not be sequentially regular.

The well-knon example of a regular space on which every continuous function is constant constructed by J. Novák in [4] yields a counter-example.

Proposition 2. A c-regular sequential Hausdorff space need not be regular.

Consider the convergence space L_{10} in [6, p.96]. The induced sequential space is a Hausdorff sequentially regular and hence, by Theorem 5, c-regular space. It is easy to verify that the space is not regular.

Notice that taking the disjoint topological sum (see Theorem 7) of the above two spaces we obtain a c-regular sequential Hausdorff space which is neither regular nor sequentially regular.

Theorem 6. A c-regular sequential T_1 space is Hausdorff.

Proof. Let Y by a c-regular sequential T_1 space and let $x, y \in Y$, $x \neq y$. Then the constant sequence $\langle x \rangle$ and y satisfy the assumption of (c) in Theorem 1 and hence can be separated by disjoint open sets.

Proposition 3. A c-regular T_1 space need not be Hausdorff.

As a counter-example there can serve the space constructed by V. Koutník in [3, Example 3].

Proposition 4. A c-regular sequential space need not be Hausdorff.

The two-point accrete space is a trivial counter-example.

Theorem 7. The class of all c-regular spaces is closed under formation of subspaces, disjoint topological sums and products.

Proof. The first two statements are self-evident. Let $Y = \prod Y_\iota$ be a product of c-regular spaces Y_ι, $\iota \in I$. Then Y is c-regular, for if $y \in Y - \overline{\cup(y_n)}$ in Y, then there is an index $\alpha \in I$ and a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $pr_\alpha(y) \in Y_\alpha - \overline{\cup(pr_\alpha(y'_n))}$ in Y_α. Since Y_α is c-regular, the assertion follows immediately.

Proposition 5. A quotient of a sequential c-regular space need not be c-regular.

Consider the convergence space L_{10} in [6, p.96]. The induced sequential space is c-regular (see Proposition 2). Let us identify the points $(\xi, 1)$, $\xi < \omega_1$, with $(1,1)$ and take the quotient space. The quotient space of a sequential space is sequential (see [2]). Since the quotient space is T_1 non Hausdorff ($(1,1)$, $(\omega_1, 1)$ cannot be separated), the proof follows from Theorem 6.

<u>Proposition 6</u>. Let G be a convergence commutative group. Then the induced sequential space need not be c-regular.

Consider the completion L_1 of the group of rational numbers constructed by J. Novák in [7]. The completion consists of the group of real numbers endowed with the sequential (Fréchet) Hausdorff topology finer than the usual one. The identical mapping on the rational numbers considered as a mapping on $\mathbb{Q}$ into L_1 can be continuously extended onto each subspace $\mathbb{Q} \cup (x)$, x irrational, but cannot be continuously extended onto $\mathbb{R}$. Thus, by Theorem 2, L_1 is not c-regular.

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Vysoká škola dopravní

katedra matematiky

01088 Žilina

Československo

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