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# NOTES ON RADICAL FILTERS OF IDEALS

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Abstract: Let $R$ be a ring and $\mathfrak{M}$ be a non-empty set of left ideals of $R$. Denote by $\mathcal{F}(\mathfrak{M})$ the radical filter generated by $\mathfrak{M}$. In this paper we give a certain characterization of $\mathcal{F}(\mathfrak{M})$.

Key words: Radical filter, hereditary torsion class, hereditary radical.

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In the following, $R$ will be an associative ring with unit and the word "module" means a unitary left $R$-module. Further, we shall denote by $R$-*mod* the category of all the $R$-modules and $\mathcal{S}(M)$ will be the set of all submodules in $M$ for any $M \in R$-*mod*. Let $\mathfrak{M} \subseteq \mathcal{S}(R)$ be a non-empty subset. Consider the following six conditions for $\mathfrak{M}$.

($F_1$) If $I \in \mathfrak{M}$, $K \in \mathcal{S}(R)$ and $I \subseteq K$, then $K \in \mathfrak{M}$.

($F_2$) If $I \in \mathfrak{M}$ and $\lambda \in R$, then $(I:\lambda) = \{\varphi \mid \varphi \in R, \varphi\lambda \in I\} \in \mathfrak{M}$.

($F_3$) If $I, K \in \mathfrak{M}$, then $I \cap K \in \mathfrak{M}$.

($F_4$) If $I, K \in \mathfrak{M}$, then $I \cdot K \in \mathfrak{M}$. and

($F_5$) If $I \in \mathfrak{M}$, $K \in \mathcal{S}(R)$, $K \subseteq I$ and $(K:\lambda) \in \mathfrak{M}\ \forall \lambda \in I$, then $K \in \mathfrak{M}$.

($F_6$) If $I \in \mathfrak{M}$, $K \in \mathcal{S}(R)$ and $(K:\lambda) \in \mathfrak{M}\ \forall \lambda \in I$, then $K \in \mathfrak{M}$.

The set $\mathfrak{M}$ is called a filter (a radical filter) if it satisfies the conditions $(F_1)$, $(F_2)$, $(F_3)$ ($(F_1)$, $(F_2)$, $(F_5)$). As it is easy to show, any radical filter satisfies all the six conditions $(F_1)$ ... $(F_6)$. Recall that there is a one-to-one correspondence between radical filters and so called hereditary radicals. A hereditary radical is an arbitrary subfunctor of the identity $\varkappa$ having the following properties:

(i) $\varkappa\left(M/\varkappa(M)\right) = 0 \ \forall M \in R\text{-}mod$ ,

(II) $\varkappa(N) = N \cap \varkappa(M) \ \forall M \in R\text{-}mod \ \forall N \in \mathcal{S}(M)$ .

If $\mathfrak{M}$ is a radical filter, then the subfunctor $\varkappa$ , given by $\varkappa(M) = \{m \mid (0 : m) \in \mathfrak{M}\}$ , is a hereditary radical. Conversely, if $\varkappa$ is a hereditary radical then $\{I \mid I \in \mathcal{S}(R), \varkappa(R/I) = R/I\}$ is a radical filter. (For the proof see e.g. [4].) A non-empty class of modules $\mathfrak{N}$ is said to be a hereditary torsion class, if it is closed under submodules, homomorphic images, extensions and direct sums. In this case, the subfunctor $\varkappa$, $\varkappa(M) = \sum_{N \in \mathcal{S}(M) \cap \mathfrak{N}} N$ is a hereditary radical. Conversely, if $\varkappa$ is a hereditary radical then $\{M \mid \varkappa(M) = M\}$ is a hereditary torsion class. Since the intersection of any set of radical filters is a radical filter, we can consider the complete lattice $\mathcal{L}(R)$ of all radical filters of the ring $R$ . Finally, denote by $\mathcal{K}(R)$ the set of all the subsets $\mathfrak{M} \subseteq \mathcal{S}(R)$ which satisfy the conditions $(F_1)$ and $(F_2)$. It is obvious that $\mathcal{K}(R)$ is a sublattice in the lattice

$2^{\mathcal{S}(R)}$ of all subsets of $\mathcal{S}(R)$ .

2. If $M \in R\text{-}mod$ and $K \in \mathcal{S}(M)$ then we denote by $\mathcal{E}^1(K,M)$ the set $\{N \mid N \in \mathcal{S}(M), K \subseteq N\}$ and by $\mathcal{E}^2(K,M)$ the set $\{N \mid N \in \mathcal{S}(M), K \subseteq N$ and $N/K$ is essential in $M/K\}$ . Further, $\mathcal{E}^3(K,M)$ will be $\mathcal{E}^2(K,M) \cup \{K\}$ .

<u>2.1. Lemma</u>. Let $M \in R\text{-}mod$ and $K, L, N \in \mathcal{S}(M)$ be such that $K \subseteq L \subseteq N$ . Then:

(i) $N \in \mathcal{E}^2(K,M)$ iff $N \cap X = K$ implies $X = K$ for arbitrary $X \in \mathcal{S}(M)$ .

(ii) $N \in \mathcal{E}^2(L,M)$ implies $N \in \mathcal{E}^2(K,M)$ .

(iii) $L \in \mathcal{E}^2(K,M)$ implies $N \in \mathcal{E}^2(K,M)$ .

<u>Proof</u>. Obvious.

Before we proceed further, let us introduce the following notation. If $M \in R\text{-}mod$ and $\emptyset \neq \mathfrak{M} \subseteq \mathcal{S}(M)$ , then by $r_{\mathfrak{M}}$ we shall mean the hereditary radical corresponding to the hereditary torsion class, which is generated by all the factor-modules $M/N$ , $N \in \mathfrak{M}$ . Further put $\mathfrak{A}(\mathfrak{M}) = \{S \mid S \in \mathcal{S}(M), \exists m \in M \setminus S$ $\forall m \in M\ \forall \lambda \in R \setminus (S:m)\ \forall N \in \mathfrak{M}\ \exists \sigma \in (N:m)$ such that $\sigma\lambda m \notin S\}$ and $\mathfrak{B}(\mathfrak{M}) = \mathcal{S}(\mathfrak{M}) \setminus \mathfrak{A}(\mathfrak{M})$ . Thus $\mathfrak{B}(\mathfrak{M}) = \{S \mid S \in \mathcal{S}(M), \forall m \in M \setminus S\ \exists m \in M\ \exists \lambda \in R \setminus (S:m)\ \exists N \in \mathfrak{M}$ such that $(N:m) \subseteq (S:\lambda m)\}$ .

<u>2.2. Lemma</u>. Let $M \in R\text{-}mod$, $A \in \mathcal{S}(M)$ and

$\emptyset \neq \mathfrak{M} \subseteq \mathcal{S}(M)$. Then $A \in \mathcal{A}(\mathfrak{M})$ iff there is $m \in$ $\in M \setminus A$ such that

$$Hom_R(B/N, {}^{Rm+A}/A) = 0$$

for all $N \in \mathfrak{M}$ and $B \in \mathcal{E}^1(N, M)$.

Proof. (i) Let $A \in \mathcal{A}(\mathfrak{M})$. Then there is $m \in M \setminus A$ such that $(N : n) \nsubseteq (A : \lambda m)$ for any $n \in M$, $N \in \mathfrak{M}$ and $\lambda \in R \setminus (A : m)$. If $\varphi : B/N \longrightarrow {}^{Rm+A}/A$ is non-zero, then $\varphi(b + N) = \rho m + A \neq 0$ for some $b \in B$ and $\rho \in R$. Hence $\rho \in R \setminus (A : m)$ and $(N : b) \subseteq (A : \rho m)$, a contradiction.

(ii) Let $A$ satisfy the condition of the lemma. If $(N : n) \subseteq$ $\subseteq (A : \lambda m)$ for some $N \in \mathfrak{M}$ and $\lambda \in R \setminus (A : m)$, then the mapping $\varphi : {}^{Rn+N}/N \longrightarrow {}^{Rm+A}/A$ defined by $\varphi(\rho n + N) = \rho\lambda m + A \ \forall \rho \in R$, is a non-zero homomorphism, a contradiction.

2.3. Lemma. Let $M \in R\text{-}mod$, $K \in \mathcal{S}(M)$ and $\emptyset \neq \mathfrak{M} \subseteq \mathcal{S}(M)$ be such that $K \in \mathcal{B}(\mathfrak{M})$. Then:

(i) $S \in \mathcal{E}^2(K, M)$, where $S/K = \kappa_{\mathfrak{M}}(M/K)$.

(ii) $\kappa_{\mathfrak{M}}(M/K) \neq 0$, provided $M \neq K$.

Proof. (i) Let $m \in M \setminus K$ be arbitrary. In view of Lemma 2.2, there is $N \in \mathfrak{M}$ and $B \in \mathcal{S}(M)$ such that $N \subseteq B$ and $Hom_R(B/N, {}^{Rm+K}/K) \neq 0$. Since $\kappa_{\mathfrak{M}}(B/N) = B/N$, $\kappa_{\mathfrak{M}}({}^{Rm+K}/K) \neq 0$. However,

$\varkappa_{\mathfrak{m}}\left({}^{Rm+K}/K\right) = {}^{Rm+K}/K \cap {}^{S}/K$ , and consequently ${}^{S}/K$ is essential in ${}^{M}/K$ .

(i) There is $m \in M \setminus K$ , and hence (by Lemma 2.2) $Hom_R\left({}^{B}/N, {}^{Rm+K}/K\right) \neq 0$ for some $N \in \mathfrak{m}$ and $B \in \mathcal{E}^1(N,M)$ . Thus $0 \neq \varkappa_{\mathfrak{m}}\left({}^{Rm+K}/K\right) \subseteq \varkappa_{\mathfrak{m}}\left({}^{M}/K\right)$ .

2.4. Lemma. Let $M \in R\text{-}mod$, $K \in \mathcal{S}(M)$ and $\emptyset \neq \mathfrak{m} \subseteq \mathcal{S}(M)$ . Then the following are equivalent:

(i) $\mathcal{E}^3(K,M) \cap \mathfrak{a}(\mathfrak{m}) \neq \emptyset$ .

(ii) $\mathcal{E}^1(K,M) \cap \mathfrak{a}(\mathfrak{m}) \neq \emptyset$ .

(iii) There are $A \in \mathcal{E}^3(K,M)$ and $S \in \mathcal{S}(M)$ such that $A \subsetneqq S$ and $\varkappa_{\mathfrak{m}}({}^{S}/A) = 0$ .

(iv) There are $A \in \mathcal{E}^1(K,M)$ and $S \in \mathcal{S}(M)$ such that $A \subsetneqq S$ and $\varkappa_{\mathfrak{m}}({}^{S}/A) = 0$ .

(v) $\varkappa_{\mathfrak{m}}({}^{M}/K) \neq {}^{M}/K$ .

Proof. (i) implies (ii) and (iii) implies (iv) trivially. (i) implies (iii). Let $A \in \mathcal{E}^3(K,M) \cap \mathfrak{a}(\mathfrak{m})$ . By Lemma 2.2, there is $m \in M \setminus A$ such that $Hom_R\left({}^{B}/N, {}^{Rm+A}/A\right) = 0$ for all $N \in \mathfrak{m}$ and $B \in \mathcal{E}^1(N, M)$ . From this, one can easily derive $\varkappa_{\mathfrak{m}}\left({}^{Rm+A}/A\right) = 0$ . Now it is sufficient to put $S = {}^{Rm+A}/A$ .

Similarly we can prove (ii) implies (iv).

(iv) implies (v). If $\varkappa_{\mathfrak{M}}({}^{M}/K) = {}^{M}/K$, then $\varkappa_{\mathfrak{M}}({}^{S}/A) = {}^{S}/A$ for all $A, S \in \mathcal{E}^1(K, M)$ such that $A \subseteq S$.

(v) implies (i). Assume, on the contrary, that $K \in \mathcal{B}(\mathfrak{M})$, and therefore, in view of Lemma 2,3, $S \in \mathcal{E}^2(K, M)$, where ${}^{S}/K = \varkappa_{\mathfrak{M}}({}^{M}/K)$. Using Lemma 2.3 again, we get $\varkappa_{\mathfrak{M}}({}^{M}/S) \neq 0$, a contradiction.

2.5. Theorem. Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid I \in \mathcal{I}(R), \mathcal{E}^1(I, R) \subseteq \mathcal{B}(\mathfrak{M})\} = \{I \mid I \in \mathcal{I}(R), \mathcal{E}^3(I, R) \subseteq \mathcal{B}(\mathfrak{M})\}$.

Proof. The theorem follows from Lemma 2.4, since $\mathcal{F}(\mathfrak{M}) = \{I \mid I \in \mathcal{I}(R), \varkappa_{\mathfrak{M}}({}^{R}/I) = {}^{R}/I\}$.

2.6. Corollary. A non-empty subset $\mathcal{R} \subseteq \mathcal{I}(R)$ is a radical filter iff it satisfies the following condition:

($F_{\dot{x}}$) If $I \in \mathcal{I}(R)$ and $\forall K \in \mathcal{E}^1(I, R)\ \forall \varkappa \in R \setminus K\ \exists \delta \in R\ \exists \lambda \in R \setminus (K : \varkappa)\ \exists L \in \mathcal{R}$ such that $(L : \delta) \subseteq (K : \lambda\varkappa)$, then $I \in \mathcal{R}$.

Proof. This corollary is only a transcription of Theorem 2.5.

For a non-empty subset $\mathfrak{M} \subseteq \mathcal{I}(R)$ put $C(\mathfrak{M}) = \{I \mid \exists \lambda \in R\ \exists K \in \mathfrak{M}$ such that $(K : \lambda) \subseteq I\}$ and $\mathcal{D}(\mathfrak{M}) = \{I \mid \forall \lambda \in R \setminus I\ \exists \varphi \in R \setminus (I : \lambda)$ such that $(I : \varphi\lambda) \in \mathfrak{M}\}$.

**2.7. Corollary.** Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid I \in \mathcal{I}(R),\ \mathcal{E}^1(I,R) \subseteq \mathcal{D}(C(\mathfrak{M}))\} =$
$= \{I \mid I \in \mathcal{I}(R),\ \mathcal{E}^3(I,R) \subseteq \mathcal{D}(C(\mathfrak{M}))\}$.

In particular, if $\mathfrak{M}$ satisfies $(F_1)$ and $(F_2)$, then
$\mathcal{F}(\mathfrak{M}) = \{I \mid \mathcal{E}^1(I,R) \subseteq \mathcal{D}(\mathfrak{M})\} = \{I \mid \mathcal{E}^3(I,R) \subseteq \mathcal{D}(\mathfrak{M})\}$.

*Proof.* The corollary follows from Theorem 2.5, since $\mathcal{B}(\mathfrak{M}) = \mathcal{D}(C(\mathfrak{M}))$, as one may check easily.

As a very easy consequence of 2.7 and 2.1 we get the following well-known result (see [3]).

**2.8. Corollary.** Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset satisfying $(F_1)$, $(F_2)$ and let $\mathcal{E}^2(0,R) \subseteq \mathfrak{M}$. Then $\mathcal{F}(\mathfrak{M}) = \mathcal{D}(\mathfrak{M})$.

**2.9. Corollary.** Let $\mathfrak{M} \subseteq \mathcal{I}(R)$ be a non-empty subset and let $\mathcal{H}(\mathfrak{M}) = \{I \mid I \in \mathcal{I}(R), \exists r \in R\setminus I\ \exists N \in \mathfrak{M}\ \exists n \in R$ such that $(N:n) \subseteq (I:r)\}$. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid \mathcal{E}^1(I,R)\setminus\{R\}\} \subseteq$
$\subseteq \mathcal{H}(\mathfrak{M})\}$.

*Proof.* (i) Let $I \in \mathcal{F}(\mathfrak{M})$, $I \neq R$. Then, by 2.6 (for $\varkappa = 1$), there are $n \in R$, $r \in R\setminus(I:1) = R\setminus I$ and $N \in \mathfrak{M}$ with $(N:n) \subseteq (I:r)$.

(ii) Let $I \in \mathcal{I}(R$ and $\{\mathcal{E}^1(I,R)\setminus\{R\}\} \subseteq \mathcal{H}(\mathfrak{M})$. Set $S/I = \varkappa_{\mathfrak{M}}(R/I)$. If $S = R$, then obviously $I \in \mathcal{F}(\mathfrak{M})$. Suppose $S \neq R$. By the hypothesis, there are $r \in R\setminus S$, $N \in \mathfrak{M}$ and $n \in R$ such that $(N:n) \subseteq (S:r)$. Thus $(S:r) \in \mathcal{F}(\mathfrak{M})$ and $r + S \in \varkappa_{\mathfrak{M}}(R/S)$, a contradiction since $\varkappa_{\mathfrak{M}}(R/S) = 0$.

**2.10. Corollary.** Let $I \in \mathcal{G}(R)$ be a two-sided ideal, $\varphi : R \to R/I$ be the canonical epimorphism and $\mathcal{R} \subseteq \mathcal{G}(R/I)$ be a radical filter. Put $\mathfrak{X} = \{K \mid K \in \mathcal{G}(R),\ I \subseteq K$ and $\varphi(K) \in \mathcal{R}\}$. Then $\varphi(L) \in \mathcal{R}$ for all $L \in \mathcal{F}(\mathfrak{X})$.

*Proof.* Let $L \in \mathcal{F}(\mathfrak{X})$ be arbitrary and $K \in \mathcal{G}(R) \setminus \{R\}$ be such that $I \subseteq K$ and $\varphi(L) \subseteq \varphi(K)$. By 2.9, there are $N \in \mathfrak{X}$, $\varkappa \in R$ and $\sigma \in R \setminus K$ with $(N : \varkappa) \subseteq (K : \sigma)$. Since $I$ is a two-sided ideal, $I \subseteq (N : \varkappa)$ and $I \subseteq (K : \sigma)$. Hence $\varphi((N:\varkappa)) = (\varphi(N) : \varphi(\varkappa)) \subseteq \varphi((K:\sigma)) = (\varphi(K) : \varphi(\sigma))$.

However, $\varphi(N) \in R$ and $\varphi(\sigma) \notin \varphi(K)$. Thus we have proved $\{\mathcal{E}^1(\varphi(L)) \setminus \{R/I\}\} \subseteq \mathcal{H}(\mathcal{R})$, and therefore $\varphi(L) \in \mathcal{R}$ (by 2.9).

**2.11. Corollary.** The lattice $\mathcal{L}(R)$ is distributive, and it is complementary iff $R$ is a semiartinian ring.

*Proof.* For $\mathcal{U}, \mathcal{V} \in \mathcal{K}(R)$ put $\mathcal{U} \varphi \mathcal{V}$ iff $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{V})$. From 2.9 it is easy to see that $\varphi$ is a congruence relation on the lattice $\mathcal{K}(R)$ and that $\mathcal{K}(R)/\varphi \cong \mathcal{L}(R)$. If, further, $\mathcal{L}(R)$ is complementary, then the radical filter $\mathcal{R}$ which is generated by all maximal left ideals possesses a complement $\mathcal{T}$, and consequently $\mathcal{R} = \mathcal{G}(R)$ (since $\mathcal{T} \cap \mathcal{R} = \{R\}$ implies $\mathcal{T} = \{R\}$). For the converse implication suppose that $R$ is semiartinian and $\mathcal{U} \in \mathcal{L}(R)$ is an element. Denote $\mathcal{V} = \{I \mid I \in \mathcal{G}(R)$ is maximal and $I \in \mathcal{U}\}$ and

$\mathcal{Z} = \{I \mid I \in \mathcal{S}(R)$ is maximal and $I \notin \mathcal{U}$ or $I = R\}$. Obviously, $\mathcal{Z}, \mathcal{V} \in \mathcal{K}(R)$. Further, since $R$ is semiartinian, $\mathcal{F}(\mathcal{V}) = \mathcal{U}$ and $\mathcal{F}(\mathcal{F}(\mathcal{U}) \cup \mathcal{F}(\mathcal{Z})) = \mathcal{S}(R)$. Finally, let $\mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{Z}) \neq \{R\}$. Then there is $I \in \mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{Z})$, $I \neq R$ is a maximal left ideal. By 2.9, $(I : \lambda) \in \mathcal{Z}$ for some $\lambda \in R \setminus I$. However, $1 = \rho\lambda + \alpha$, where $\rho \in R$ and $\alpha \in I$ are suitable, and so $I = (I : \rho\lambda) = ((I : \lambda) : \rho) \in \mathcal{Z}$. Thus $I \in \mathcal{Z} \cap \mathcal{V}$, a contradiction.

Let us note here that the preceding corollary was already proved before in [1] for the case of commuative noetherian rings.

3. In this paragraph we generalize some results from [2] to get a characterization of $\mathcal{F}(\mathcal{M})$, where $\mathcal{M}$ is a countable set of two-sided ideals. Let $\mathcal{M} = \{I_1, I_2, \dots\}$ be a countable subsystem of $\mathcal{S}(R)$. A sequence $\{\lambda_1, \lambda_2, \dots\}$ of elements from $R$ will be called $\mathcal{M}$-regular if the set $\{i \mid \lambda_i \in I_j\}$ is infinite for any $j = 1, 2, \dots$. Denote by $\mathcal{R}(\mathcal{M})$ the set of all the $\mathcal{M}$-regular sequences and put $\mathcal{G}(\mathcal{M}) = \{I \mid \forall \{\lambda_1, \lambda_2, \dots\} \in \mathcal{R}(\mathcal{M}) \forall \rho \in R\ \exists n \geq 1$ such that $\lambda_n \dots \lambda_1 \rho \in I\}$.

<u>3.1. Theorem</u>. Let $\mathcal{M} = \{I_1, I_2, \dots\}$ be a countable subsystem of $\mathcal{S}(R)$. Then:

(i) $\mathcal{G}(\mathcal{M})$ is a radical filter.

(ii) $\mathcal{G}(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$.

(iii) $\mathcal{G}(\mathcal{M}) = \mathcal{F}(\mathcal{M})$ provided every ideal from $\mathcal{M}$ is two-sided.

Proof. (i) The condition $(F_1)$ is obvious. Now $(F_2)$. Let $I \in \mathcal{G}(\mathfrak{M})$ and $\sigma \in R$ be arbitrary. If $\{\lambda_1, \lambda_2, \ldots\} \in \mathcal{U}(\mathfrak{M})$ and $\varphi \in R$, then (by the hypothesis) there is $n \geq 1$ such that $\lambda_n \ldots \lambda_1 \varphi \sigma \in I$, i.e. $\lambda_n \ldots \lambda_1 \varphi \in (I : \sigma)$. Finally $(F_6)$. Let $I \in \mathcal{S}(R)$, $K \in \mathcal{G}(\mathfrak{M})$ and $(I : \varkappa) \in \mathcal{G}(\mathfrak{M})$ for each $\varkappa \in K$. Given $\{\lambda_1, \lambda_2, \ldots\} \in \mathcal{U}(\mathfrak{M})$ and $\varphi \in R$, there is $n \geq 1$ with $\lambda_n \ldots \lambda_1 \varphi \in K$. However, the sequence $\{\lambda_{n+1}, \lambda_{n+2}, \ldots\}$ is also $\mathfrak{M}$-regular and $(I : \lambda_n \ldots \lambda_1 \varphi) \in \mathcal{G}(\mathfrak{M})$. Hence there is $m \geq 1$ such that $\lambda_{n+m} \ldots \lambda_{n+1} \cdot 1 \in (I : \lambda_n \ldots \lambda_1 \varphi)$ and so $\lambda_{n+m} \ldots \lambda_n \ldots \lambda_1 \varphi \in I$.

(ii) Suppose, on the contrary, that there exists $I \in \mathcal{G}(\mathfrak{M})$, $I \notin \mathcal{F}(\mathfrak{M})$. Hence (by $(F_6)$) there is $\lambda_1 \in I_1$ such that $(I : \lambda_1) \notin \mathcal{F}(\mathfrak{M})$. Further, $I_1 \cap I_2 \in \mathcal{F}(\mathfrak{M})$ and therefore there is $\lambda_2 \in I_1 \cap I_2$ such that $(I : \lambda_2 \lambda_1) = ((I : \lambda_1) : \lambda_2) \notin \mathcal{F}(\mathfrak{M})$. Repeating this argument, we get a sequence $\{\lambda_1, \lambda_2, \ldots\}$ having the following properties:

($\alpha$) $\lambda_j \in I_1 \cap I_2 \cap \ldots \cap I_j$ for every $j = 1, 2, \ldots$,

($\beta$) $(I : \lambda_j \ldots \lambda_1) \notin \mathcal{F}(\mathfrak{M})$ for every $j = 1, 2, \ldots$.

From ($\alpha$) we see that $\{\lambda_1, \lambda_2, \ldots\}$ is an $\mathfrak{M}$-regular sequence. Hence, by the hypothesis, $\lambda_n \ldots \lambda_1 . 1 \in I$ for some $n \geq 1$, and consequently $(I : \lambda_n \ldots \lambda_1) = R$, which is a contradiction with ($\beta$).

(iii) Obvious, since $I_j \in \mathcal{G}(\mathfrak{M})$ whenever $I_j$ is a two-sided ideal.

3.2. Corollary. Let $\mathfrak{M} = \{I_1, \ldots, I_n\}$ be a finite set of two-sided ideals. Then $\mathcal{F}(\mathfrak{M}) = \{I \mid \forall \lambda_1, \lambda_2, \ldots \in I_1 \cap \ldots \cap I_n \ \exists m \geq 1$ such that $\lambda_m \ldots \lambda_1 \in I\}$.

Proof. Denote by $\mathcal{I}$ the set defined above. From 3.1 it is obvious that $\mathcal{F}(\mathcal{M}) \subseteq \mathcal{I}$. In order to prove the converse inclusion we need only to observe the following fact. If $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathcal{M})$, then there exist $1 \leq \leq \ell_1 < \ell_2 < \ell_3 < \dots$ such that $\lambda_{\ell_j} . \lambda_{\ell_j - 1} \dots \lambda_{\ell_{j-1}} \in I_1 \cap \dots \cap I_n$ for all $j = 1, 2, \dots$.

3.3. Corollary. Let $\mathcal{M}$ be a finite set of two-sided ideals. Then $0 \in \mathcal{F}(\mathcal{M})$ iff $\bigcap_{I \in \mathcal{M}} I$ is right $T$-nilpotent.

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