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A NOTE ON OPERATOR CONVERGENCE FOR SEMIGROUPS

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Abstract: Let $\Gamma = \{T_t ; t > 0\}$ be a strongly continuous semigroup of linear contractions on a Hilbert space H and let $f \in H$. It is proved that if weak-$\lim_{t\to\infty} T_t f = f_\infty$ for some $f_\infty \in H$ then strong-$\lim_{n\to\infty} \int_0^\infty a_n(t) T_t f \, dt = f_\infty$ for any sequence $\{a_n\}$ of nonnegative Lebesgue integrable functions on $(0,\infty)$ such that $\int_0^\infty a_n(t)\,dt = 1$ for each n and $\lim_{n\to\infty} \| a_n \|_\infty = 0$.

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Let H be a Hilbert space and let $\Gamma = \{T_t ; t > 0\}$ be a semigroup of linear contractions on H, i.e., each T_t is a bounded linear operator from H to H with $\|T_t\| \le 1$ and $T_t T_s = T_{t+s}$ for all $t, s > 0$. In this note we shall assume that Γ is strongly continuous. This means that $\lim_{t\to s} \| T_t f - T_s f \| = 0$ for all $f \in H$ and all $s > 0$. It follows that for any complex-valued Lebesgue integrable function $a(t)$ on $(0,\infty)$, the vector-valued function $t \longrightarrow a(t) T_t f$ on $(0,\infty)$ is also Lebesgue integrable. The purpose of this note is to prove the following result, which is a continuous version of the Blum-Hanson theorem [1] (see also [2]).

Theorem. Let $f \in H$ and $\text{weak-}\lim_{t\to\infty} T_t f = f_\infty$ for some $f_\infty \in H$. Then for any sequence $\{a_n\}$ of nonnegative Lebesgue integrable functions on $(0,\infty)$ such that $\int_0^\infty a_n(t)\,dt = 1$ for each n and $\lim_{n\to\infty} \|a_n\|_\infty = 0$, we have $\text{strong-}\lim_{n\to\infty} \int_0^\infty a_n(t)\, T_t f\, dt = f_\infty$.

Proof. Since $T_t f_\infty = f_\infty$ for all $t > 0$, we may and will assume without loss of generality that $f_\infty = 0$. Since $\|T_t\| \leq 1$ for all $t > 0$, $\lim_{t\to\infty} \|T_t f\|$ exists. Thus for a given $\varepsilon > 0$, there exists a positive number M such that

$$|\langle T_t f, f\rangle| < \varepsilon \quad \text{and} \quad \|T_t f\|^2 - \|T_{t+s} f\|^2 < \varepsilon^2$$

for all $t > M$ and all $s > 0$. It then follows that

$$\|T_s^* T_{t+s} f - T_t f\|^2 = \|T_s^* T_{t+s} f\|^2 + \|T_t f\|^2 - 2\|T_{t+s} f\|^2$$

$$\leq \|T_t f\|^2 - \|T_{t+s} f\|^2 < \varepsilon^2$$

for all $t > M$ and all $s > 0$, and hence

$$|\langle T_{t+s} f, T_s f\rangle| \leq |\langle T_{t+s} f, T_s f\rangle - \langle T_t f, f\rangle| + |\langle T_t f, f\rangle|$$

$$\leq |\langle T_s^* T_{t+s} f - T_t f, f\rangle| + \varepsilon$$

$$\leq \|T_s^* T_{t+s} f - T_t f\|\,\|f\| + \varepsilon$$

$$\leq \varepsilon(\|f\| + 1)$$

for all $t > M$ and all $s > 0$. Therefore

$$
\begin{aligned}
\| \int_0^\infty a_n(t) T_t f \, dt \|^2 &= \langle \int_0^\infty a_n(t) T_t f \, dt, \int_0^\infty a_n(t) T_t f \, dt \rangle \\
&= \int_0^\infty \int_0^\infty \langle a_n(t) T_t f, a_n(s) T_s f \rangle \, dt \, ds \\
&\leq \int_0^\infty \int_0^\infty a_n(t) a_n(s) | \langle T_t f, T_s f \rangle | \, dt \, ds \\
&\leq \| f \|^2 \| a_n \|_\infty (2M) \int_0^\infty a_n(t) \, dt \\
&\quad + \varepsilon (\| f \| + 1) \int_0^\infty a_n(t) \, dt \int_0^\infty a_n(s) \, ds \\
&= \| f \|^2 \| a_n \|_\infty (2M) + \varepsilon (\| f \| + 1) ,
\end{aligned}
$$

where the fourth inequality follows from Fubini's theorem. Since $\lim_{n \to \infty} \| a_n \|_\infty = 0$ by Hypothesis, this completes the proof.

R e f e r e n c e s

[1] BLUM J.R., HANSON D.L.: On the mean ergodic theorem for subsequences, Bull.Amer.Math.Soc.66(1960), 308-311.

[2] JONES L., KUFTINEC V.: A note on the Blum-Hanson theorem, Proc.Amer.Math.Soc.30(1971),202-203.

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