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ON A QUESTION OF PULTR REGARDING CATEGORIES OF STRUCTURES

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Abstract: It is known that every constructive structure can be realized as a structure based on a power (under composition) of the contravariant power-set functor. It is proved here that one can use the covariant one instead.

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Aleš Pultr has given a definition which allows one to describe models of higher order theories in terms of first-order structures defined in the range of a functor from $\mathbf{Set}$ to $\mathbf{Set}$. This suggests the question: which functors generate structures comparable with those of ordinary nth order logic (for some n)? Pultr has given a partial answer by finding a class of categories of models that can be realized in $S((P^-)^n \circ V_A)$, the category of all models $(X, \mathfrak{U})$ whose structure $\mathfrak{U}$ consists of a distinguished subset of $((P^-)^n \circ V_A)(X)$, where P^- is the usual contravariant power set functor and V_A is a sum of the identity functor and a constant functor. The present paper gives a similar partial answer by showing that these same categories can be realized in $S((P^+)^n \circ V_A)$, where P^+

is the usual covariant power set functor. As with Pultr's work, if one is willing to allow infinite powers of P^+, then the class of functors involved can be enlarged by taking limits and colimits over small categories.

When not specified, the terminology is as in [1]. **Set** denotes the category of sets and functions. For any function $f: X \longrightarrow Y$, let $f^\vee$ equal $(P^-)(f): \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$, and let $f^\sim$ ambiguously represent $(P^+)^k(f): \mathcal{P}^k(X) \longrightarrow \mathcal{P}^k(Y)$.

1 Lemma: $S((P^-)^2)$ is realizable in $S((P^+)^4)$; $(P^-)^2$ is majorized by $(P^+)^5$.

Proof. For any $\mathcal{U} \subseteq \mathcal{P}(X)$ and $A \subseteq X$, define A to be $\mathcal{U}$-substantial iff $\forall U \subseteq X$, $U \in \mathcal{U}$ iff $U \cap A \in \mathcal{U}$.

Step I: For any function $f: X \rightarrow Y$ and $\mathcal{U} \subseteq \mathcal{P}(X)$, if A is $\mathcal{U}$-substantial, then $f[A]$ is $f^{\vee\vee}(\mathcal{U})$-substantial. Since $f^{\vee\vee}(\mathcal{U}) = \{V \subseteq Y: f^\vee(V) \in \mathcal{U}\}$, we have that $\forall V \subseteq Y$, $V \cap f[A] \in f^{\vee\vee}(\mathcal{U})$ iff $f^\vee(V \cap f[A]) \in \mathcal{U}$; but $f^\vee(V \cap f[A]) = f^\vee(V) \cap f^\vee(f[A])$, and $f^\vee(V) \cap f^\vee(f[A]) \in \mathcal{U}$ iff $f^\vee(V) \cap A \in \mathcal{U}$, iff $f^\vee(V) \in \mathcal{U}$ iff $V \in f^{\vee\vee}(\mathcal{U})$. Hence $f[A]$ is $f^{\vee\vee}(\mathcal{U})$-substantial.

Define a functor $\mathbf{R}: \mathbf{Set} \rightarrow \mathbf{Set}$ as follows: for any set X, $\mathbf{R}(X)$ is the set of all pairs $\{\mathfrak{X}, \mathbb{Q}\}$ such that

i) $\mathfrak{X} \subseteq \{\{U\}: U \subseteq X\}$,

ii) $\emptyset \in \bigcup\mathbb{Q}$, $\mathbb{Q} \subseteq \{\{Q_1, Q_2\}: Q_1, Q_2 \subseteq X\}$ and $\mathbb{Q} \supseteq \{\{Q_1, Q_2\}: Q_1 \neq Q_2$ and $Q_1, Q_2 \in \bigcup\mathbb{Q}\}$,

iii) $\bigcup\bigcup\mathfrak{X} \subseteq \bigcup\bigcup\mathbb{Q}$;

for any map $f : X \longrightarrow Y$ let $\mathbf{R}(f) = (P^{+})^4(f)$. By nonstandard convention, we shall consider phrases such as

" $\{\mathcal{X}, \mathcal{Q}\} \in \mathbf{R}(X)$ " to abbreviate " $\{\mathcal{X}, \mathcal{Q}\} \in \mathbf{R}(X)$, $\mathcal{X}$ satisfies (i), and $\mathcal{Q}$ satisfies (ii)".

Step II: If $f : X \to Y$, $\{\mathcal{X}, \mathcal{Q}\} \in \mathbf{R}(X)$, $\{\mathcal{Y}, \mathcal{R}\} \in \mathbf{R}(Y)$, and $f^{\sim}(\{\mathcal{X}, \mathcal{Q}\}) = \{\mathcal{Y}, \mathcal{R}\}$, then $f^{\sim}(\mathcal{X}) = \mathcal{Y}$ and $f^{\sim}(\mathcal{Q}) = \mathcal{R}$. Suppose not; then $f^{\sim}(\mathcal{Q}) = \mathcal{Y}$ and $f^{\sim}(\mathcal{X}) = \mathcal{R}$. Now if $\bigcup\bigcup \mathcal{Q}$ were non-empty, $f^{\sim}(\mathcal{Q})$ would contain a nontrivial pair of the form $\{\emptyset, f[Q]\}$. But $\mathcal{Y}$ contains only singletons. Hence $\mathcal{Q} = \{\{\emptyset\}\}$ since $\emptyset \in \bigcup \mathcal{Q}$. Consequently $f^{\sim}(\mathcal{Q}) = \{\{\emptyset\}\}$. Similarly, $\bigcup\bigcup f^{\sim}(\mathcal{X}) = \bigcup\bigcup \mathcal{R}$ must be empty, so that $\mathcal{R} = \{\{\emptyset\}\} = \mathcal{X}$. Hence $f^{\sim}(\mathcal{X}) = \mathcal{Y}$ and $f^{\sim}(\mathcal{Q}) = \mathcal{R}$.

For any $\{\mathcal{X}, \mathcal{Q}\} \in \mathbf{R}(X)$, define $\mathcal{Q}$ to be significant iff $\forall \{Q_1, Q_2\} \in \mathcal{Q}$, $Q_1 \cap Q_2 = \emptyset$.

Step III: It is easy to see that given $f : X \longrightarrow Y$ and $\{\mathcal{X}, \mathcal{Q}\} \in \mathbf{R}(X)$, $f^{\sim}(\mathcal{Q})$ is significant iff $\mathcal{Q}$ is significant and $\forall Q_1, Q_2 \in \bigcup \mathcal{Q}$, $Q_1 \neq Q_2$ implies $f[Q_1] \cap f[Q_2] = \emptyset$.

A realization of $\mathcal{S}((P^{-})^2)$ in $\mathcal{S}(\mathbf{R})$ can now be given as follows: for each X and $\mathcal{U} \subseteq \mathcal{P}^2(X)$, let $\mathcal{U}^*$ be the set of all $\{\mathcal{X}, \mathcal{Q}\} \in \mathbf{R}(X)$ such that if $\mathcal{Q}$ is significant, then for some $\mathcal{U} \in \mathcal{U}$, $\bigcup\bigcup \mathcal{Q}$ is $\mathcal{U}$-substantial and $\bigcup \mathcal{X} = \{U \in \mathcal{U} : \exists \mathcal{a} \subseteq \bigcup \mathcal{Q}, U = \bigcup \mathcal{a}\}$. Let $f : X \to Y$, $\mathcal{U} \subseteq \mathcal{P}^2(X)$, and $\mathcal{V} \subseteq \mathcal{P}^2(Y)$ be arbitrary.

Step IV: If $\mathbf{R}(f)[\mathcal{U}^*] \subseteq \mathcal{V}^*$, then $f^{\vee\vee}[\mathcal{U}] \subseteq \mathcal{V}$. Pick $\mathcal{U} \in \mathcal{U}$. Let $\mathcal{Q}$ be the set of all pairs $\{f^{\vee}(A), f^{\vee}(B)\}$

such that $A, B \subseteq Y$, $A \cap B = \emptyset$, and $card\, A, card\, B \leq 1$. Let $\mathfrak{X} = \{\{U\}: U \in \mathcal{U}$ and $\exists \mathcal{a} \subseteq \bigcup \mathcal{Q}, U = \bigcup \mathcal{a}\}$. Then $\{\mathfrak{X}, \mathcal{Q}\} \in \mathcal{U}^*$, and thus $f^{\sim}(\{\mathfrak{X}, \mathcal{Q}\}) \in \mathcal{V}^*$, $f^{\sim}(\mathcal{Q})$ is clearly significant, and thus we may choose $\mathcal{V} \in V$ so that $\bigcup\bigcup f^{\sim}(\mathcal{Q})$ is $\mathcal{V}$-substantial and $\bigcup f^{\sim}(\mathfrak{X}) = \{V \in \mathcal{V}: \exists \mathcal{B} \subseteq \bigcup f^{\sim}(\mathcal{Q}), V = \bigcup \mathcal{B}\}$. We need to show $\mathcal{V} = f^{\vee\vee}(\mathcal{U})$. From the choice of $\mathcal{V}$ and the definition of $\mathcal{Q}$, it is clear that $\bigcup f^{\sim}(\mathfrak{X}) = \{V \in \mathcal{V}: V \subseteq f[X]\}$. Hence $\bigcup f^{\sim}(\mathfrak{X}) = \mathcal{V} | f[X]$ since $f[X]$ is $\mathcal{V}$-substantial. From the definitions of $\mathfrak{X}$ and $\mathcal{Q}$, it is clear that

$$\bigcup f^{\sim}(\mathfrak{X}) = \{V \subseteq f[X]: f^{\vee}(V) \in \mathcal{U}\}$$

$$= \{V \in f^{\vee\vee}(\mathcal{U}); V \subseteq f[X]\} .$$

Hence $\bigcup f^{\sim}(\mathfrak{X}) = f^{\vee\vee}(\mathcal{U}) | f[X]$ since $f[X]$ is $f^{\vee\vee}(\mathcal{U})$-substantial, so that $\mathcal{V} | f[X] = f^{\vee\vee}(\mathcal{U}) | f[X]$. But then $\mathcal{V} = f^{\vee\vee}(\mathcal{U})$ by substantialness. Therefore $f^{\vee\vee}[\mathcal{U}] \subseteq \mathcal{V}$.

Step V: If $f^{\vee\vee}[\mathcal{U}] \subseteq \mathcal{V}$, then $R(f)[\mathcal{U}^*] \subseteq \mathcal{V}^*$. Pick $\{\mathfrak{X}, \mathcal{Q}\} \in \mathcal{U}^*$. If $f^{\sim}(\mathcal{Q})$ isn't significant, then $R(f)(\{\mathfrak{X}, \mathcal{Q}\}) = \{f^{\sim}(\mathfrak{X}), f^{\sim}(\mathcal{Q})\} \in \mathcal{V}^*$. If $f^{\sim}(\mathcal{Q})$ is significant, then so is $\mathcal{Q}$, and for some $\mathcal{U} \in U$, $\bigcup\bigcup \mathcal{Q}$ is $\mathcal{U}$-substantial and $\bigcup \mathfrak{X} = \{U \in \mathcal{U}: \exists \mathcal{a} \subseteq \bigcup \mathcal{Q}, U = \bigcup \mathcal{a}\}$. But then $f^{\sim}(\bigcup\bigcup \mathcal{Q})$ is $f^{\vee\vee}(\mathcal{U})$-substantial and $f^{\vee\vee}(\mathcal{U}) \in \mathcal{V}$. To see that $f^{\sim}(\{\mathfrak{X}, \mathcal{Q}\}) \in \mathcal{V}^*$, we need to show that

$$\bigcup f^{\sim}(\mathfrak{X}) = \{V \in f^{\vee\vee}(\mathcal{U}): \exists \mathcal{a} \subseteq \bigcup \mathcal{Q}, V = \bigcup f^{\sim}(\mathcal{a})\} .$$

Pick $V \in \bigcup f^{\sim}(\mathfrak{X})$; then for some $U \in \mathcal{U}$ and $\mathcal{a} \subseteq \bigcup \mathcal{Q}$, $U = \bigcup \mathcal{a}$ and $f[U] = V$. We have $f^{\vee}(f[U]) \cap \bigcup\bigcup \mathcal{Q} = U$, since if not, there would be some $\mathcal{Q}_1 \in \mathcal{a}$ and $\mathcal{Q}_2 \in \bigcup\mathcal{Q} - \mathcal{a}$ such that $f[\mathcal{Q}_1] \cap f[\mathcal{Q}_2] \neq \emptyset$, in which case $f^{\sim}(\mathcal{Q})$ wouldn't be significant. Consequently, $f^{\vee}(f[\mathcal{U}]) \in \mathcal{U}$ since $\bigcup\bigcup\mathcal{Q}$ is $\mathcal{U}$-substantial. Hence $f[\mathcal{U}] \in f^{\vee\vee}(\mathcal{U})$. Conversely, if $V \in f^{\vee\vee}(\mathcal{U})$ and for some $\mathcal{a} \subseteq \bigcup\mathcal{Q}$, $V = \bigcup f^{\sim}(\mathcal{a})$, then $f^{\vee}(V) \cap \bigcup\bigcup\mathcal{Q} = \bigcup\mathcal{a}$ again since $f^{\sim}(\mathcal{Q})$ would otherwise not be significant. Since $f^{\vee}(V) \in \mathcal{U}$ and $\bigcup\bigcup\mathcal{Q}$ is $\mathcal{U}$-substantial, $f^{\vee}(V) \cap \bigcup\bigcup\mathcal{Q} \in \mathcal{U}$. Hence $f^{\vee}(V) \cap \bigcup\bigcup\mathcal{Q} \in \bigcup\mathfrak{X}$, and

$$f[f^{\vee}(V) \cap \bigcup\bigcup\mathcal{Q}] = f[\bigcup\mathcal{a}] = V \in \bigcup f^{\sim}(\mathfrak{X}) .$$

Therefore $f^{\sim}(\{\mathfrak{X},\mathcal{Q}\}) \in \mathcal{V}^*$, as required.

We have just shown that the map $\mathcal{U} \mapsto \mathcal{U}^*$ induces a realization of $S((P^-)^2)$ in $S(R)$. Since for each structure $\mathcal{U} \subseteq \mathcal{P}^2(X)$, $\mathcal{U}^* \subseteq (P^+)^4(X)$, the same construction may be considered as a realization of $S((P^-)^2)$ in $S((P^+)^4)$. Using a similar construction, we can now show that $(P^+)^5$ majorizes $(P^-)^2$. For each set X, each $\mathcal{U} \subseteq \mathcal{P}(X)$, and each $\mathcal{U}$-substantial $A \subseteq X$, let $\mathcal{U}_A$ be the set of all $\{\mathfrak{X},\mathcal{Q}\} \in R(A)$ such that $\bigcup\bigcup\mathcal{Q} = A$ and if $\mathcal{Q}$ is significant, then $\bigcup\mathfrak{X} = \{U \in \mathcal{U} : \exists\mathcal{a} \subseteq \bigcup\mathcal{Q}, U = \bigcup\mathcal{a}\}$. Define a functor $E : \mathit{Set} \longrightarrow \mathit{Set}$ as follows: for each set X, let $E(X) = \{\mathcal{U}_A : \mathcal{U} \subseteq \mathcal{P}(X)$ and A is $\mathcal{U}$-substantial$\}$; for each function $f : X \longrightarrow Y$ and $\mathcal{U}_A \in E(X)$, let $E(f)(\mathcal{U}_A) = (P^+)^5(f)$. E is in fact a functor, as a result of the following

Step VI: For any given $f : X \longrightarrow Y$ and $\mathcal{U}_A \in E(X)$, $E(f)(\mathcal{U}_A) = f^{\vee\vee}(\mathcal{U})_{f[A]}$. The argument of step V shows that $E(f)(\mathcal{U}_A) \subseteq f^{\vee\vee}(\mathcal{U}_A)_{f[A]}$. Now pick $\{\mathcal{Y},\mathcal{R}\} \in f^{\vee\vee}(\mathcal{U})_{f[A]}$. Let $\mathfrak{X} = \{\{f^{\vee}[V] \cap A\} : V \in \bigcup\mathcal{Y}\}$, and let $\mathcal{Q} = \{\{f^{\vee}(R_1 \cap A), f^{\vee}(R_2 \cap A)\} : \{R_1, R_2\} \in \mathcal{R}\}$. Clearly, $f^{\sim}(\{\mathfrak{X},\mathcal{Q}\}) = \{\mathcal{Y},\mathcal{R}\}$ and $\bigcup\bigcup\mathfrak{X} \subseteq \bigcup\bigcup\mathcal{Q} = A$, so that $\{\mathfrak{X},\mathcal{Q}\} \in \mathbb{R}(A)$. If $\mathcal{R}$ isn't significant, neither is $\mathcal{Q}$, and thus $\{\mathfrak{X},\mathcal{Q}\} \in \mathcal{U}_A$. Assume $\mathcal{R}$ is significant; then so is $\mathcal{Q}$. To see that $\{\mathfrak{X},\mathcal{Q}\} \in \mathcal{U}_A$, we need to show that $\bigcup\mathfrak{X} = \{U \in \mathcal{U} : \exists \mathcal{a} \subseteq \bigcup\mathcal{Q} , U = \bigcup\mathcal{a}\}$. First pick $U \in \bigcup\mathfrak{X}$; then $f[U] \in \bigcup\mathcal{Y}$, so that for some $\mathcal{B} \subseteq \bigcup\mathcal{R}$, $f[U] = \bigcup\mathcal{B}$ and $f[U] \in f^{\vee\vee}(\mathcal{U})$. But if $\mathcal{a} = \{f^{\vee}[B] \cap A : B \in \mathcal{B}\}$, then $\mathcal{a} \subseteq \bigcup\mathcal{Q}$, $U = f^{\vee}(f[U]) \cap A = \bigcup\mathcal{a}$, and $U \in \mathcal{U}$ since A is $\mathcal{U}$-substantial and $f^{\vee}(f[U]) \in \mathcal{U}$, since $f[U] \in f^{\vee\vee}(\mathcal{U})$. Conversely, if $U \in \mathcal{U}$, $\mathcal{a} \subseteq \bigcup\mathcal{Q}$, and $U = \bigcup\mathcal{a}$, then $f[U] = \bigcup f^{\sim}(\mathcal{a})$ with $f^{\sim}(\mathcal{a}) \subseteq \bigcup\mathcal{R}$. Moreover, $f^{\vee}(f[U]) \cap A = U \in \mathcal{U}$, so that $f^{\vee}(f[U]) \in \mathcal{U}$ and $f[U] \in f^{\vee\vee}(\mathcal{U})$, so that $f[U] \in \bigcup\mathcal{Y}$. But then $U = f^{\vee}(f[U]) \cap A \in \bigcup\mathfrak{X}$. Therefore $\{\mathfrak{X},\mathcal{Q}\} \in \mathcal{U}_A$.

For each set X , let φ_X be the inclusion map from $E(X)$ to $(P^+)^5(X)$. φ is clearly a monotransformation from E to $(P^+)^5$. Now define an epitransformation ψ from E to $(P^-)^2$ as follows: $\forall \mathcal{U}_A \in E(X)$, $\psi_X(\mathcal{U}_A) = \mathcal{U}$. Each ψ_X is well-defined since each $\mathcal{U}_A$ contains a pair $\{\mathfrak{X},\mathcal{Q}\}$ such that $\bigcup\mathfrak{X} = \mathcal{U}|A$ (just let $\mathcal{Q} = \{\{Q_1, Q_2\} : Q_1, Q_2 \subseteq A , Q_1 \cap Q_2 = \emptyset$, and $card\, Q_1$, $card\, Q_2 \leq 1\}$) . Each ψ_X is clearly onto; to see that ψ is a natural transformation from E to $(P^-)^2$, pick $f : X \longrightarrow Y$ and $\mathcal{U}_A \in E(X)$; then $(P^-)^2(f)(\psi_X(\mathcal{U}_A)) = f^{\vee\vee}(\mathcal{U}) = \psi_Y(f^{\vee\vee}(\mathcal{U})_{f[A]})$

$$= \psi_Y(E(f))(\mathcal{U}_A)) .$$

Therefore $(P^+)^5$ majorizes $(P^-)^2$.

2 **Theorem.** If $G_1, \ldots, G_n$ are constructively majorizable functors and $\Delta_1, \ldots, \Delta_n$ are types, then $S((G_1, \Delta_1), \ldots, (G_n, \Delta_n))$ is realizable in $S((P^+)^k \circ V_A)$ for some set A and natural number k .

Proof. The numbered theorems which will be referred to are those of [1]. By Theorem 6.5, $S((G_1, \Delta_1), \ldots, (G_n, \Delta_n))$ is realizable in $S((P^-)^k \circ V_M)$ for some number k and set M . If k is odd, then $S((P^-)^k \circ V_M)$ is realizable in $S((P^-)^{k+1} \circ V_M)$ by Theorem 1.5. Hence $S((G_1, \Delta_1), \ldots, (G_n, \Delta_n))$ is realizable in some $S((P^-)^{2n} \circ V_M)$. By Corollary 3.7 and the above lemma, $(P^-)^{2n} \circ V_M$ is majorized by $(P^+)^{5n} \circ V_M$. Hence by Theorem 6.1, $S((P^-)^{2n} \circ V_M)$ is realizable in $S((P^+)^{5n} \circ V_M)$.

Problem: Characterize the class of all categories $S(F)$ which can be realized in some $S((P^+)^k \circ V_A)$ (or, equivalently, $S((P^-)^k \circ V_A)$). Characterize the class of all categories $S(F, \Delta)$ which can be realized in some $S((P^+)^k, \Gamma)$ (equivalently, in $S((P^-)^k, \Gamma)$).

The above theorem may be extended to the infinite case with the help of the following result.

3 Lemma. For each monotransformation $\tau : I \to (P^+)^n$ there is an $m \geq n$ and a monotransformation $\theta : (P^+)^n \to (P^+)^m$ such that $\theta\tau = \xi^m$, where $\xi : I \to P^+$ is the unique monotransformation.

Proof: First we need some facts about natural transformations from I to $(P^+)^n$. By Remark 2.9 of [2], the natural transformations from I to $(P^+)^n$ are in 1-1 correspondence with the elements of $(P^+)^n(\{\emptyset\})$, and for any set $A \in (P^+)^n(\{\emptyset\})$, we may let $\tau_{n,A}$ be the transformation such that for each set X and $x \in X$, $\tau_{n,A,X}(x) = (P^+)^n(\varepsilon_x)(A)$, where $\varepsilon_x : \{\emptyset\} \to X$ is given by $\varepsilon_x(\emptyset) = x$. Since $\tau_{n,A,X}$ doesn't depend on X in a significant way, we will usually drop this third subscript. Notice that if $A \in (P^+)^{n+1}(\{\emptyset\})$, then

$$\tau_{n+1,A}(x) = (P^+)^{n+1}(\varepsilon_x)(A) = \{(P^+)^n(\varepsilon_x)(a) : a \in A\} = \{\tau_{n,a}(x) : a \in A\}.$$

1) The following are equivalent:

a) $\tau_{n,A}$ is a monotransformation

b) $\mathrm{rank}\, A = n$ (where $\mathrm{rank}\, A$ is inductively defined as the smallest ordinal greater than $\mathrm{rank}\, a$, for all $a \in A$).

c) $\forall x$, $\bigcup^n \tau_{n,A}(x) = x$, where for any set S, $\bigcup^0 S = S$ and $\bigcup^{n+1}(S) = \bigcup\{\bigcup^n s : s \in S\}$.

d) $\exists x$, $\bigcup^n \tau_{n,A}(x) \neq \emptyset$.

Proof: The only element of $(P^+)^0(\{\emptyset\})$ is $\emptyset$, and so $\tau_{0,\emptyset} : I \to I$ is the identity transformation; $\tau_{0,\emptyset}$ clearly satisfies the four conditions. By induction, assume for $n \geq 0$ that the four conditions are equivalent. Pick $A \in (P^+)^{n+1}(\{\emptyset\})$. Then $\mathrm{rank}\, A = n+1$ iff for some $a \in A$, $\mathrm{rank}\, a = n$, in which case $\tau_{n,a}$ would satisfy the four conditions. Thus if $\mathrm{rank}\, A = n+1$, then

$$\begin{aligned}
\textstyle\bigcup^{n+1} \tau_{n+1,A}(x) &= \textstyle\bigcup^{n+1}\{\tau_{n,a}(x) : a \in A\} \\
&= \textstyle\bigcup\{\bigcup^n \tau_{n,a}(x) : a \in A\} \\
&\left\{\begin{array}{l} = \bigcup\{x\}, \text{ if } \forall a \in A,\ \mathrm{rank}\, a = n \\ = \bigcup\{x, \emptyset\}, \text{ if } \exists a \in A,\ \mathrm{rank}\, a < n \end{array}\right. \\
&= x,
\end{aligned}$$

and so the four conditions hold. But if $rank\, A < n+1$, then

$$\bigcup^{n+1}\tau_{n+1,A}(x) = \bigcup\{\bigcup^{n}\tau_{n,a}(x) : a \in A\} = \bigcup\{\emptyset\} = \emptyset ,$$

and they don't hold.

For any set X, let π_X be the unique map from X to $\{\emptyset\}$. For each natural number k and $C \in (P^+)^k(X)$, define the k-type of C to be $(P^+)^k(\pi_X)(C)$. Notice that a set $A \in (P^+)^{k+1}(\{\emptyset\})$ is the $k+1$-type of $\mathcal{C} \in (P^+)^{k+1}(X)$ iff A is the set of k-types of elements of $\mathcal{C}$. We will need the following properties of natural transformations from $(P^+)^j$ to $(P^+)^k$:

2) Suppose that $A \in (P^+)^k(\{\emptyset\})$ and $rank\, A < k$. Then for any set Y, $A \in (P^+)^k(Y)$, as can be easily seen by induction on the rank of A. Consequently the constant transformation γ from $(P^+)^j$ to $(P^+)^k$, given by $\forall X$, $\forall C \in (P^+)^j(X)$, $\gamma_X(C) = A$ is natural.

3) If $C \in (P^+)^j(X)$ and $f : X \to Y$, then $(P^+)^j(C)$ has the same j-type as C since

$$(P^+)^j(\pi_Y)((P^+)^j(f)(C)) = (P^+)^j(\pi_Y f)(C)$$
$$= (P^+)^j(\pi_X)(C) .$$

From this fact, it follows immediately that given $\varphi, \psi :$ $: (P^+)^j \to (P^+)^k$ and $\Delta \subseteq (P^+)^j(\{\emptyset\})$, one can define a natural transformation $\theta : (P^+)^j \to (P^+)^k$ by $\forall X, \forall C \in (P^+)^j(X)$,

$$\theta_X(C) = \begin{cases} \varphi_X(C), & \text{if the } j\text{-type of } C \text{ is in } \Delta \\ \psi_X(C), & \text{otherwise.} \end{cases}$$

4) The same fact guarantees that if for each $a \in \Delta$, we

choose some $\theta_a : (P^+)^j \longrightarrow (P^+)^k$, and define $\varphi : (P^+)^{j+1} \longrightarrow$ $\longrightarrow (P^+)^{k+1}$ by $\forall X, \forall \mathscr{C} \in (P^+)^{j+1}(X)$, $\varphi_X(\mathscr{C}) = \{\theta_{aX}(C) : C \in \mathscr{C}, a \in \Delta$, and a is the j-type of $C\}$, then φ is also a natural transformation. Notice that if each $\theta_{aX}(C)$ is of k-type $\xi^k(\emptyset)$, then either $\varphi_X(\mathscr{C})$ is of $k+1$-type $\xi^{k+1}(\emptyset)$, or, possibly, $\varphi_X(\mathscr{C}) = \emptyset$.

5) Given natural transformations $\varphi_1, \dots, \varphi_p$ from $(P^+)^j$ to $(P^+)^k$, we can define a product transformation $\varphi_1 \times \dots \times \varphi_p : (P^+)^j \longrightarrow (P^+)^{k+p}$ as follows: inductively define $\langle x \rangle = \{x\}$, and
$\langle x_1, \dots, x_{m+1} \rangle = \{\langle x_1, \dots, x_m \rangle, \langle x_1, \dots, x_m \rangle \cup \xi^m(x_{m+1})\}$. It is easy to see that $\bigcap \langle x_1, \dots, x_{m+1} \rangle = \langle x_1, \dots, x_m \rangle$ and (by induction) that $\bigcup^m \langle x_1, \dots, x_{m+1} \rangle = \{x_1, \dots, x_{m+1}\}$, so that this is an acceptable convention for m-tuples. Also, if $x_1, \dots, x_p \in X$, then $\langle x_1, \dots, x_p \rangle \in (P^+)^p(X)$; hence if $C \in (P^+)^j(X)$, then $\langle \varphi_1(C), \dots, \varphi_p(C) \rangle = \varphi_1 \times \dots \times \varphi_p(C) \in$ $\in (P^+)^{k+p}(X)$. Notice that if $\langle D_1, \dots, D_p \rangle$ are of k-type $\xi^k(\emptyset)$, then $\langle D_1, \dots, D_p \rangle$ is of $k+p$-type $\xi^{k+p}(\emptyset)$.

We can now find the required $\theta : (P^+)^n \longrightarrow (P^+)^m$ as follows: for $m = 0$ the only monotransformation from I to $(P^+)^n$ is the identity. For $m = 1$, the only one is ξ itself. In either case we may let θ be the identity on $(P^+)^m$. Notice that if $a \in (P^+)^m(\{\emptyset\})$, then for each set X and $x \in X$, $\tau_{m,a}$ is characterized by the fact that the m-type of $\tau_{m,a}(x)$ is a, since

$$(P^+)^m(\pi_X)(\tau_{m,a}(x)) = \tau_{m,a}(\pi_X(x)) = \tau_{m,a}(\varphi) = a .$$

Our inductive assumption will, accordingly, be that for

$n \geq 1$, there is a $k \geq n$ such that for each monotransformation $\tau_{n,a} : I \longrightarrow (P^+)^n$, there is a monotransformation $\theta_a : (P^+)^n \longrightarrow (P^+)^k$ such that whenever $C \in (P^+)^n$ is of n-type a, $\theta_a(C)$ is of k-type $\xi^k(\emptyset)$. We then have, in particular that $\forall x$, $\tau_{n,a}(x)$ is of n-type a, and $\theta_a \tau_{n,a}(x)$ is of k-type $\xi^k(\emptyset)$, so that $\theta_a \tau_{n,a} = \tau_{k,\xi^k(\emptyset)} = \xi^k$. Let $\tau_{n+1,A} : I \longrightarrow (P^+)^{n+1}$ be any fixed monotransformation. Let $A = \{a_1, \dots, a_p\} \cup \{b_1, \dots, b_q\}$ be an indexing of A such that $a_1, \dots, a_p$ are the elements of A of rank n. For each a_i, let θ_i be a monotransformation from $(P^+)^n$ to $(P^+)^k$ satisfying the induction hypothesis. Define $\varphi_i : (P^+)^{n+1} \longrightarrow (P^+)^{k+1}$ by $\forall X$, $\forall \mathcal{C} \in (P^+)^{n+1}(X)$

$$\varphi_{iX}(\mathcal{C}) = \{\theta_{iX}(C) : C \in \mathcal{C} \text{ and } C \text{ is of } n\text{-type } a_i\}.$$

Let $\theta : (P^+)^{n+1} \longrightarrow (P^+)^{k+p+1}$ be given by $\forall \mathcal{C} \in (P^+)^{n+1}(X)$, $\theta_X(\mathcal{C}) = \varphi_1 \times \dots \times \varphi_m(\mathcal{C})$, if $\mathcal{C}$ is of $n+1$-type A, and $\theta_X(\mathcal{C}) = \{\xi^{p+k-n-1}(\mathcal{C}), \emptyset\}$ otherwise. The φ_i are natural by (4), and θ is natural by (3), (5), and (4) and (2).

To see that if $\mathcal{C}$ is of $n+1$-type A, then $\theta_X(\mathcal{C})$ is of $p+k+1$-type $\xi^{k+p+1}(\emptyset)$, notice first that $\{a_1, \dots, a_p\}$ is nonempty by (1) since $\tau_{n,A}$ is a monotransformation. Each element of each $\varphi_{iX}(\mathcal{C})$ is of k-type $\xi^k(\emptyset)$ by the inductive assumption. Hence each element of $\varphi_1 \times \dots \times \varphi_p(\mathcal{C})$ is of $k+p$-type $\xi^{k+p}(\emptyset)$, so that $\varphi_1 \times \dots \times \varphi_p(\mathcal{C})$ is of $k+p+1$-type $\xi^{k+p+1}(\emptyset)$.

Finally, each θ_X is mono: let $\theta_X(\mathcal{C})$ be given.

$\mathcal{C}$ may be recovered as follows: if $\emptyset \in \theta_X(\mathcal{C})$, then $\mathcal{C} = \bigcup^{p+k-m}\theta_X(\mathcal{C})$. Assume $\emptyset \notin \theta_X(\mathcal{C})$. Then $\mathcal{C}$ is of $m+1$-type A. Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$, where $\mathcal{C}_1$ is the set of elements of $\mathcal{C}$ of rank less than m, and $\mathcal{C}_0$ is the rest. We know that $(P^+)^{m+1}(\pi_X)(\mathcal{C}) = A = \{a_1, \dots, a_p\} \cup \{b_1, \dots, b_q\}$. By an easy induction we have that $\forall C \in (P^+)^m(X)$, rank $C \geq m$ iff $rank\,(P^+)^m(\pi_X)(C) = m$, and that if rank $C < m$, then $(P^+)^m(\pi_X)(C) = C$. Consequently, $\mathcal{C}_1 = \{b_1, \dots, b_q\}$, and $\{a_1, \dots, a_p\}$ is the $m+1$-type of $\mathcal{C}_0$. For each a_i, let η_i be a left inverse function for θ_{iX}; clearly,

$\mathcal{C}_0 = \{\eta_i(D) : D$ is the i^{th} element of some p-tuple in $\theta_X(\mathcal{C})\}$.

As it stands, the number $m_A = k + p + 1$ depends on A, since p does. However, a uniform $m = max\{m_A : A \in (P^+)^{m+1}(X)\}$ is easily obtained by composing θ with ξ^{m-m_A}. This completes the induction.

4 <u>Theorem</u>. Let F_ι $(\iota \in \Gamma)$ be TB-functors (in the sense of [2]), and Δ_ι $(\iota \in \Gamma)$ types. Then there is an ordinal α and a set A such that

$S((F_\iota, \Delta)_{\iota \in \Gamma}) \Rrightarrow S((P^+)^\alpha \circ V_A)$.

<u>Proof</u>. Let $\lambda : I \to (P^-)^2$ be the monotransformation given by $\forall X, \forall x \in X$, $\lambda_X(x) = \{A \subseteq X : x \in A\}$. Define $\mu : I \to E$ by $\forall X, \forall x \in X$, $\mu_X(x) = \lambda_X(x)_{\{x\}} = \{\{\mathfrak{X}, \mathfrak{Q}\} \in \mathfrak{R}(\{x\}) : \bigcup\bigcup \mathfrak{Q} = \{x\}$, and if $\mathfrak{Q}$ is significant, then $\bigcup\mathfrak{X} = \{\{x\}\}\}$. The condition that $\bigcup\bigcup\mathfrak{X} \subseteq \bigcup\bigcup\mathfrak{Q} = \{x\}$

forces $\mu_X(x)$ to be independent of X, and a moment's thought shows that μ is a monotransformation. As at the end of Lemma 1, let $\varphi : E \rightarrow (P^+)^5$ be the monotransformation given by the equation $\varphi_X(\mathcal{U}_A) = \mathcal{U}_A$, and let $\psi : E \rightarrow (P^-)^2$ be the epitransformation given by $\psi_X(\mathcal{U}_A) = \mathcal{U}$. Then $\psi\mu = \lambda$. Finally, for some m bigger than 5, we may let $\theta : (P^+)^5 \longrightarrow (P^+)^m$ be a monotransformation such that $\theta\varphi\mu = \xi^m$.

We need to show that any functor of the form $((P^-)^2)^\beta$ is majorized by some $(P^+, \xi)^\alpha$. Let α be a limit ordinal larger than β. Then $((P^-)^2)^\beta < ((P^-)^2)^\alpha$ by Lemma 3.7 of [2]. The equations $\psi\mu = \lambda$ and $\theta\varphi\mu = \xi^m$, and Lemma 2.8 of [2] show that

$$(((P^-)^2, \lambda)^\alpha < (E, \mu)^\alpha < ((P^+)^5, \varphi\mu)^\alpha < ((P^+)^m, \xi^m)^\alpha .$$

But by Lemma 2.4 of [2], $((P^+)^m, \xi^m)^\alpha \simeq (P^+, \xi)^\alpha$, since the first colimit is just being taken over a subsequence of the second. Now by Theorem 3.7 of [1], we have $((P^-)^2, \lambda)^\beta \circ V_A < (P^+, \xi)^\alpha \circ V_A$, for any set A, and thus by Theorem 6.1 of [1], $S(((P^-)^2, \lambda)^\beta \circ V_A) \Rightarrow S((P^+, \xi)^\alpha \circ V_A)$. Finally, let $S(F_\iota, \Delta_\iota)_{\iota \in T}$ be as in the statement of the theorem. Then by Theorem 4.2 of [2], $S((F_\iota, \Delta_\iota)_{\iota \in T}) \Rightarrow$ $\Rightarrow S(((P^-)^2, \lambda)^\beta \circ V_A)$, for some ordinal β and set A and the theorem follows.

R e f e r e n c e s

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