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ON WEAK HOMOTOPY

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Abstract: If the definition of homotopy is weakened by using the cross-product instead of the usual cartesian product of spaces, all connected polyhedra become contractible.

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The cross-product $X \otimes Y$ (the space obtained from the cartesian product of the underlying sets by the condition that $f: X \otimes Y \longrightarrow Z$ is continuous iff it is continuous in each variable) is well-known to be a tensor product in the category of topological spaces. Thus, we can base on it a notion similar to homotopy - we will call it weak homotopy or W-homotopy - defined as follows:

$f, g : X \longrightarrow Y$ are said to be W-homotopic if there is an $h : X \otimes I \longrightarrow Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

Thus, W-homotopy is a weaker equivalence than the nor-

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mal one. In this paper we are going to show that it is actually much weaker: e.g. all connected polyhedra are W-homotopically trivial.

It is evident that every W-homotopically trivial space has to be arcwise connected. The converse is probably not true, but we do not have a counterexample. I am indebted to prof. Pultr, who suggested this problem, and who gave me valuable help.

1. Conventions and notations

Throughout this paper the circle is considered as the interval $[0,1]$, with identified endpoints. The closed (open) unit-interval will be denoted by $I (J)$. The closed unit-ball (sphere) in the n-dimensional Euclidean space R^n will be denoted by B_n (S_n). The polyhedra will always be connected, and they are supposed to be embedded in a suitable Euclidean space. The points of this Euclidean space are sometimes considered as vectors - in order to simplify the notation. For every point $p \in R^n$, we define $U(p) = \vec{p} / \|\vec{p}\|$. Given two pointed spaces (X, x_0) and (Y, y_0), $(X, x_0) \# (Y, y_0)$ is the topological space, obtained from $X \times X$ identifying the points (x, y) with $x = x_0$ or $y = y_0$ (with the quotient-topology).

2.

Proposition 1. The products of W-homotopically trivial spaces are W-homotopically trivial.

Proof. Given a family $(X_a)_a$ of W-homotopically trivial spaces with homotopy-functions f_a, consider the following diagram:

$$\begin{array}{ccc}
\left(\prod_{a \in A} X_a\right) \otimes I & \xrightarrow{\;p_b \otimes id_I\;} & X_b \otimes I \\
\Big\downarrow {\scriptstyle \prod_{a \in A} f_a} & & \Big\downarrow {\scriptstyle f_b} \\
\left(\prod_{a \in A} X_a\right) & \xrightarrow[\;p_b\;]{} & X_b
\end{array}$$

where $\prod_{a \in A} f_a$ is defined in the following way: $\prod_{a \in A} f_a((x_a)_a, t) = (f_a(x_a, t))_a$. This function is continuous.

Proposition 2. The long line is W-homotopically trivial.

Proof. Let $L = \{(x,y) \mid x \in \mathbb{R}, y \in [0,1[\,\}$ be endowed with the lexicographical order, and the associated order-topology. The function $h : L \otimes I \longrightarrow L$; $h((x,y),t) = (xt, yt)$, is continuous, and L is W-homotopically trivial.

Proposition 3. The circle is W-homotopically trivial.

Proof. Consider $h : S \otimes I \longrightarrow S$ defined by:

$$\begin{aligned} h(\vartheta, t) &= \vartheta^{1/t} && \text{if } t \neq 0 \\ &= 0 && \text{if } t = 0 \;. \end{aligned}$$

Clearly, h is continuous.

Corollary. Every torus is W-homotopically trivial.

3.

Suspension

Proposition. The suspension of an arbitrary space is W-homotopically trivial.

Proof. Let (X, x_0) be an arbitrary pointed space. Define $h : ((X, x_0) \times (S_1, 0)) \otimes I \longrightarrow (X, x_0) \times (S_1, 0)$ by

$$h(((x, \vartheta), t)) = \begin{cases} (x, \vartheta^{1/t}) & \text{if } t \neq 0 \\ (x, 0) & \text{if } t = 0 \, . \end{cases}$$

Let $g : (X, x_0) \times (S_1, 0) \longrightarrow (x, x_0) \# (S_1, 0)$ be the natural quotient-mapping. h is usually not continuous, but $g \circ h$ is. The commutativity of the diagram

$$\begin{array}{ccc} ((X, x_0) \times (S_1, 0)) \otimes I & \xrightarrow{\quad g \circ h \quad} & \\ \big\downarrow & & (X, x_0) \# (S_1, 0) \\ ((X, x_0) \# (S_1, 0)) \otimes I & \xrightarrow{\quad h^* \quad} & \end{array}$$

defines uniquely a continuous mapping h^* (because $g \otimes id$ is a quotient mapping).

Corollary. Every sphere is W-homotopically trivial.

4. *Polyhedra*

Proposition 1. All one-dimensional connected polyhedra are W-homotopically trivial. If x_0 is an arbitrary vertex of the polyhedron P, then the homotopy functions can be chosen in such a way that $\forall t \in I,\ f(x_0, t) = x_0$.

Proof. The proposition is trivial for all one-dimensional polyhedra with at most two vertices. Suppose it is proved for all one-dimensional polyhedra with at most $n - 1$ vertices, $n \geq 3$. Let P be an arbitrary but fixed polyhedron with m vertices, embedded in a suitable $\mathbb{R}^{n}$, and suppose all segments of P have length 1. Choose an arbitrary vertex x_0 of P, denote the vertices of P by $(x_i)_{0 \leq i \leq m-1}$.

The segments $[x_i, x_j] \in P$, x_i and $x_j \neq x_0$, form at most $m-1$ maximal connected one-dimensional polyhedra P'_k; $k \leq i_p \leq n-1$; $P'_k \cap P'_{k'} = \emptyset$ if $k \neq k'$. Choose $x_{j_k} \in P'_k$ such that $[x_{j_k}, x_0] \in P$, $\forall k \in i_p$. Consider the polyhedra P_k, consisting of the vertices of P'_k and x_0, and all the segments in P between these vertices. By induction, the P'_k are W-homotopically trivial, and there exist continuous functions $f_k : P'_k \otimes I \longrightarrow P'_k$ such that

$$f_k(x, 1) = x, \quad \forall x \in P'_k$$

$$f_k(x, 0) = x_{j_k}, \quad \forall x \in P'_k$$

$$f_k(x_{j_k}, t) = x_{j_k}, \quad \forall t \in I \quad .$$

We will define the homotopy functions g_k on the polyhedra P_k. Suppose k fixed for the time being.

1) Consider the segment $[x_0, x_{j_k}]$.

Define $g_k(x,t) = t \cdot \overrightarrow{x_0 x}$ if $x \in [x_0, x_{j_k}]$.

2) Consider the polyhedron P'_k.

Define $d_k : P'_k \times P'_k \longrightarrow \mathbb{R}_+$ by

$$d_k(y, y') = \inf \{ \sum_{a=1}^{m-1} \| x_a x_{a-1} \| \,\big|\, x_1 = y, x_m = y', x_a \in P'_k, [x_a, x_{a+1}] \subset P'_k \} .$$

A) $t = 1$

put $g_k(x,1) = f_k(x,1) = x$

B) $t \neq 1$

a) if $d_k(f_k(x,t), x_{j_k}) \geq 1/2$

put $g_k(x,t) = f_k(x,t)$

b) if $1/4 \leq d_k(f_k(x,t), x_{j_k}) \leq 1/2$

put $g_k(x,t) = 2(\overrightarrow{x_{j_k} f_k(x,t)} - (1/4).\overrightarrow{x_{j_k} x_j})$, where

$f_k(x,t) \in [x_{j_k}, x_j]$ and x_j is uniquely determined

c) if $0 \leq d_k(f_k(x,t), x_{j_k}) \leq 1/4$

put $g_k(x,t) = 4.d_k(f_k(x,t), x_{j_k}).\overrightarrow{f(x_{j_k},t)\, x_{j_k}}$.

3) Consider the segments

$[x_0, x_j]$, $x_j \in P'_k$; $[x_0, x_j] \in P$, $j \neq j_k$.

Define $h_k : (P_k - P'_k) \otimes I \longrightarrow R_+$ by

$$h_k(x,t) = \begin{cases} \|\overrightarrow{x_j x}\|^{1/t} & \text{if } t \neq 0 \text{ , and } x \in [x_0, x_j] \\ 0 & \text{if } t = 0 \end{cases}$$

if $t = 1$, put $g_k(x,1) = x$

if $t \neq 1$

a) if $1/2 \leq h_k(x,t)$

put $g_k(x,t) = h_k(x,t).\overrightarrow{x_j x}$; $x \in [x_0, x_j]$

b) if $1/4 \leq h_k(x,t) \leq 1/2$

put $g_k(x,t) = 2(h_k(x,t) - 1/2).\overrightarrow{x_j x}$,

c) if $0 \leq h_k(x,t) \leq 1/4$,

i) $t=0$

put $g_k(x,0)=x_0$

ii) $t\neq 0$

put $q_{j,k}: J\longrightarrow R_+$ by

$q_{j,k}(t)=\max\{\sum_{i=1}^{m-1}\|\overrightarrow{g_k(x_j,t_i)\,g_k(x_j,t_{i+1})}\|\mid m\in N,\ (t_i)_i$

partitions of $[t,1[\,,\ [g_k(x_j,t_i),g_k(x_j,t_{i+1})]\subset P\}$.

Define $r_{j,k}: Im(q_{j,k})\longrightarrow P_k$ by

$$r_{j,k}(q_{j,k}(t))=g_k(x_j,t)\ ,$$

define $s_{j,k,t}:[x_j,x_{t,j}]\longrightarrow[0,q_{j,k}]$ by

$s_{j,k,t}(x)=q_{j,k}(t).(1-4h_k(x,t))$ if $x\in[x_j,x_0]$

and where $x_{t,j}$ is that point on $[x_j,x_0]$ such that $h_k(x_{t,j},t)=1/4$. Define $g_k(x,t)=r_{j,k}\circ s_{j,k,t}(x)$ if $x\in[x_j,x_0]$.

4) <u>The polyhedron</u> P . Define $g(x,t)=g_k(x,t)$ if $x\in P_k$. It is clear from the construction that $g:$ $:P\otimes I\longrightarrow P$ is a continuous function such that $g(-,1)=id_P$, $g(-,0)=x_0$.

<u>Proposition 2</u>. All connected polyhedra are W-homotopically trivial.

<u>Proof</u>. The theorem is proved for all one-dimensional polyhedra, suppose it is proved for all d-dimensional ones, with $d\leq m-1, m\geq 2$. Let P be an arbitrary fixed m-dimensional polyhedron embedded in a suitable R^n. P' is the $(m-1)$-dimensional skeleton of P , with a homotopy function g' .

A) Define $g(x,t) = g'(x,t)$ for $x \in P'$.

B) 1) There exist $f_k : B_n \longrightarrow R^n$, $1 \leq k \leq m$, such that

$f_k(B_m) \subset P$, $\forall k \leq m$

$f_k | B_m$ is a homeomorphism onto the image

$f_k | S_n \subset P'$

$f_k(B_m) \cap f_{k'}(B_m) \subset P'$; $k \neq k'$

$\bigcup_{k=1}^{m} f_k(B_m) \cup P' = P$.

2) If B_m is the unit-ball, define $h' : B_m \times I \longrightarrow B_m$ as follows:

a) $h'((0,0,\dots,0),t) = (1-t,0,\dots,0)$

b) $y \neq (0,0,\dots,0)$: $h'(y,t) \in [(1-t,0,\dots,0), U(y)]$

and

$$\| \vec{y} \| = \frac{\| \overrightarrow{h'(y,t)} - \overrightarrow{(1-t,0,\dots,0)} \|}{\| \overrightarrow{(1-t,0,\dots,0)} - \overrightarrow{U(y)} \|}$$

take an $h : B_m \otimes J \longrightarrow B_m$ such that

$h((0,0,\dots,0),t) = h'((0,0,\dots,0),t)$

$h(y,t) \in [(1-t,0,\dots,0), U(y)]$, $y \neq (0,0,\dots,0)$

and

$$\frac{\| \overrightarrow{h(y,t)} - \overrightarrow{U(y)} \|}{\| \overrightarrow{(1-t,0,\dots,0)} - \overrightarrow{U(y)} \|} = \left(\frac{\| \overrightarrow{h'(y,t)} - \overrightarrow{U(y)} \|}{\| \overrightarrow{(1-t,0,\dots,0)} - \overrightarrow{U(y)} \|} \right)^{1/t}$$

3) If $x \in P - P'$, then $\exists! k \leq m$ such that $x \in f_k(B_m)$. Define the functions $h_k : f_k(B_m) \otimes J \longrightarrow f_k(B_m)$ by

$$h_k(x,t) = f_k \circ h((f_k^{-1}(x),t)) .$$

4) $x \in P'$.

Define $q_x : I \longrightarrow \mathbb{R}$ by

$$q_x(t) = \sup \{ \sum_{i=1}^{u-1} \| \overrightarrow{g'(x,t_i)\, g'(x,t_{i+1})} \|$$

where $(t_i)_i$ are partitions of $[t,1] \}$.

Define $r_x : Im(q_x) \longrightarrow Im(g'(x,-)) \subset P'$ by $r_x(q_x(t)) = g'(x,t)$.

5) $x \in f_k(B_n) - P'$; k fixed.

a) Put $g_k(x,1) = x$ and $g_k(x,0) = x_0$, where $x_0 = g'(-,0)$

b) $t \in J$.

<u>Notation</u>:

$$v(y,t) = \| \overrightarrow{(1-t,0,\dots,0)} - \overrightarrow{U(y)} \| ,\ y \in B_n ,\ y \neq (0,0,\dots,0)$$

$$\mu_k(x,t) = d(h(f_k^{-1}(x),t),\ U(f_k^{-1}(x))) .$$

Let $A_{x,t,k}$ and $B_{x,t,k}$ be the points on the segment $[(1-t,0,\dots,0),\ U(f_k^{-1}(x))]$ such that

$$\| \overrightarrow{A_{x,t,k}} - \overrightarrow{(1-t,0,\dots,0)} \| = v(f_k^{-1}(x),t)/2$$

$$\| \overrightarrow{B_{x,t,k}} - \overrightarrow{(1-\theta,0,\dots,0)} \| = 3v(f_k^{-1}(x),t)/4$$

1) If $v(f_k^{-1}(x),t)/2 \leq \mu_k(x,t)$ put

$$g_k(x,t) = h_k(x,t) .$$

2) If $v(f_k^{-1}(x),t)/4 \leq \mu_k(x,t) \leq v(f_k^{-1}(x),t)/2$

define the linear functions

$$v_{x,t,k} : [A_{x,t,k}, B_{x,t,k}] \longrightarrow [A_{x,t,k}, U(f_k^{-1}(x))]$$

such that

$$v_{x,t,k}(A_{x,t,k}) = A_{x,t,k}$$

$$v_{x,t,k}(B_{x,t,k}) = U(f_k^{-1}(x))$$

define $g_k(x,t) = f_k \circ v_{x,t,k} \circ h((f_k^{-1}(x), t))$.

3) If $0 \leq \mu_k(x,t) \leq v(f_k^{-1}(x), t)/4$

define $s_{y,t,k} : [U(y), B_{x,t,k}] \longrightarrow [0, q_\alpha(t)]$, where $x = f_k(y)$ and $\alpha = f_k(U(y))$, to be the linear functions such that

$$s_{y,t,k}(B_{x,t,k}) = 0$$

$$s_{y,t,k}(U(y)) = q_\alpha(t) ,$$

define $g_k(x,t) = x_\alpha \circ s_{y,t,k}(x)$, where $x = f_k(y)$, $\alpha = f_k(U(y))$.

4) $x \in P - P'$

put $g(x,t) = g_k(x,t)$ if $x \in f_k(B_n)$.

The function $g : P \otimes I \longrightarrow P$ is continuous.

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