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P R E R A D I C A L S

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Abstract: The purpose of this paper is to provide an essential background for the theory of preradicals in modules.

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Every mathematician working in module theory and in the torsion theories, in particular, feels that a lot of talking about torsion theories can be extended to preradicals, in general. On the other hand, no background for the theory of preradicals (except for scattered quotations) has ever been provided, as far as we know. Hence our aim is to bring such a background, ready for further use. The authors have been investigating the properties of preradicals more deeply and some of their results have already been submitted for publication ([1],[2],[3]). The theory of preradicals appears to be the real know-how in the theory of modules and rings. In particular, it seems to be an ideal tool for dualization problems.

Now, let us introduce a few definitions.

All rings will be associative and with identity and will be denoted by R . A preradical $\varkappa$ for R-*mod* (the category of unitary left R-modules) is any subfunctor of the identity functor. If $\varkappa$ is a preradical then $T_\varkappa = \{M; \varkappa(M) = M\}$ and $F_\varkappa = \{M; \varkappa(M) = 0\}$. The modules from $T_\varkappa$ ($F_\varkappa$) are called $\varkappa$-torsion ($\varkappa$-torsionfree) modules . If $\varkappa(M) \in T_\varkappa$ ($M/\varkappa(M) \in F_\varkappa$), for all $M \in R$-*mod* , then we shall say that $\varkappa$ is idempotent ($\varkappa$ is a radical). Further, the symbols $\coprod_I$ and $M^{(I)}$ are used for the direct sum and the symbol $\prod_I$ for the direct product of modules. If $\varkappa, \sigma$ are preradicals then $\varkappa \subseteq \sigma$ if $\varkappa(M) \subseteq \sigma(M)$ for every module M . A class of modules is called hereditary (cohereditary) if it is closed under submodules and isomorphic images (under epimorphic images).

If a prospective reader will find some of the proofs too short, it is due to the fact that obvious parts are omitted.

<u>Proposition 1</u>. Let $\varkappa$ be a preradical, $M \in R$-*mod* and $N \subseteq M$ be a submodule. Then:

(i) $\varkappa(N) \subseteq N \cap \varkappa(M)$.

(ii) $(\varkappa(M) + N)/N \subseteq \varkappa(M/N)$.

(iii) If $\varkappa(M/N) = 0$ then $\varkappa(M) \subseteq N$.

(iv) If $\varkappa(N) = N$ then $N \subseteq \varkappa(M)$.

<u>Proposition 2</u>. Let $\varkappa$ be a preradical and M_i, $i \in I$,

be a family of modules. Then $r(\coprod_{i\in I} M_i) = \coprod_{i\in I} r(M_i)$ and $r(\prod_{i\in I} M_i) \subseteq \prod_{i\in I} r(M_i)$.

Proposition 3. Let r be a preradical. Then:

(i) T_r is a cohereditary class closed under arbitrary direct sums.

(ii) F_r is a hereditary class closed under arbitrary direct products.

(iii) $Hom_R(T,F) = 0$ for all $T \in T_r$ and $F \in F_r$.

(iv) $T_r \cap F_r = 0$

(v) If M_i , $i \in I$ is a family of submodules of a module M such that $M_i \in T_r$, for all $i \in I$, then $\sum_{i\in I} M_i \in T_r$.

(vi) If M_i , $i \in I$, is a family of submodules of a module M such that $M/M_i \in F_r$, for all $i \in I$, then $M/\bigcap_{i\in I} M_i \in F_r$.

Proposition 4. Let r be a preradical and $M \in R\text{-}mod$. Then:

(i) $r(M)$ is a characteristic submodule of M .

(ii) If $M \in R\text{-}mod\text{-}R$ then $r(M) \in R\text{-}mod\text{-}R$.

(iii) If M is free then $r(M) \in R\text{-}mod\text{-}R$.

(iv) $r(R)$ is a twosided ideal.

(v) $r(R) . M \subseteq r(M)$.

(vi) If M is projective then $\kappa(M) = \kappa(R).M$.

Proof. (ii) The right R -multiplication on M is a left R -endomorphism of M .

(v) Let $m \in M$ be arbitrary. The mapping $f: R \to M$, given by $a \mapsto am$, is a homomorphism, and consequently $\kappa(R).m = f(\kappa(R)) \subseteq \kappa(M)$.

(vi) There is a free module F such that $F = M \oplus N$. We can suppose that $F = R^{(I)}$, for some index set I . Then

$$\kappa(F) = \kappa(R^{(I)}) = (\kappa(R))^{(I)} = (\kappa(R).R)^{(I)} = \kappa(R).R^{(I)}$$

by Proposition 2. Further, $\kappa(M) \oplus \kappa(N) = \kappa(F) =$

$$= \kappa(R).F = \kappa(R).(M \oplus N) = \kappa(R).M \oplus \kappa(R).N .$$

However, $\kappa(R).M \subseteq \kappa(M)$, $\kappa(R).N \subseteq \kappa(N)$, and therefore $\kappa(R).M = \kappa(M)$.

Proposition 5. Let κ be a preradical, and for every $M \in R\text{-}mod$ let $\bar{\kappa}(M) = \Sigma N$, where N runs through all the κ -torsion submodules of M . Then:

(i) $\bar{\kappa}$ is an idempotent preradical, $\bar{\kappa} \subseteq \kappa$ and $T_\kappa = T_{\bar{\kappa}}$.

(ii) If ρ is an idempotent preradical and $\rho \subseteq \kappa$, then $\rho \subseteq \bar{\kappa}$. Hence $\bar{\kappa}$ is the largest idempotent preradical contained in κ .

Proof. (i) is obvious from Proposition 3.

(ii) Since $\rho \subseteq \kappa$, $T_\rho \subseteq T_\kappa$, and hence $\rho(M) \in T_\kappa$, for all $M \in R\text{-}mod$. Thus $\rho(M) \subseteq \bar{\kappa}(M)$.

Proposition 6. Let κ be a preradical, and for every $M \in R\text{-}mod$ let $\tilde{\kappa}(M) = \bigcap N$, where N runs through all the submodules $N \subseteq M$ with $M/N \in F_\kappa$. Then:

(i) $\tilde{\kappa}$ is a radical, $\kappa \subseteq \tilde{\kappa}$ and $F_\kappa = F_{\tilde{\kappa}}$.

(ii) If s is a radical and $\kappa \subseteq s$, then $\tilde{\kappa} \subseteq s$. Hence $\tilde{\kappa}$ is the least radical containing κ.

Proof. The proof is similar to that of Proposition 5.

Proposition 7. Let κ be a preradical. Then the following are equivalent:

(i) If $M \in R\text{-}mod$ and $N \subseteq M$ is a submodule such that $\kappa(M) \subseteq N (N \subseteq \kappa(M))$, then $\kappa(N) = \kappa(M) (\kappa(M/N) = \kappa(M)/N)$.

(ii) κ is idempotent (κ is a radical).

(iii) $\kappa = \bar{\kappa}$ ($\kappa = \tilde{\kappa}$).

Definition. Let κ be a preradical. The preradical $\bar{\kappa}$ ($\tilde{\kappa}$) is called the idempotent core (the radical closure) of κ

Proposition 8. Let κ be an idempotent preradical. Then:

(i) $F \in F_\kappa$ iff $Hom_R(T, F) = 0$ for all $T \in T_\kappa$.

(ii) F_κ is closed under extensions.

Proof. (i) According to Proposition 3 we have only to prove the sufficiency. But if $Hom_R(T, F) = 0$, for each $T \in T_\kappa$, then $\kappa(F) = 0$ since $\kappa(F) \in T_\kappa$.

(ii) is an easy consequence of (i).

Proposition 9. Let r be a radical. Then:

(i) $T \in T_r$ iff $Hom_R(T, F) = 0$ for all $F \in F_r$.

(ii) T_r is closed under extensions.

Proof. The proof is similar to that of the preceding proposition.

Theorem 10. Let r be a preradical. Then the following are equivalent:

(i) r is an idempotent radical.

(ii) For each $M \in R\text{-}mod$ there exists a uniquely determined (up to an isomorphism) exact sequence $0 \to T \to M \to F \to 0$ with $T \in T_r$ and $F \in F_r$.

(iii) For each $M \in R\text{-}mod$ there is an exact sequence $0 \to T \to M \to F \to 0$ with $T \in T_r$ and $F \in F_r$.

(iv) r is idempotent and $T_r = T_{\tilde{r}}$.

(v) r is idempotent and T_r is closed under extensions.

(vi) $F_r = F_{\bar{r}}$ and T_r is closed under extensions.

(vii) $F_r = F_{\bar{r}}$ and $T_r = T_{\tilde{r}}$.

(viii) $T_r = T_{\tilde{r}}$ and F_r is closed under extensions.

(ix) r is a radical and $F_r = F_{\bar{r}}$.

(x) r is a radical and F_r is closed under extensions.

(xi) $r = \bar{r} = \tilde{r}$.

Proof. (i) $\Longrightarrow$ (ii). Obviously $0 \to r(M) \to (M) \to M/r(M) \to 0$ is the desired sequence.

(ii) $\Longrightarrow$ (iii) and (xi) $\Longrightarrow$ (i) trivially.

(iii) $\Longrightarrow$ (xi). Let $M \in R\text{-}mod$ and $0 \to T \xrightarrow{f} M \to F \to 0$ be an exact sequence with $T \in T_\kappa$ and $F \in F_\kappa$. Then $f(T) \in T_\kappa$ and $M/f(T) \in F_\kappa$, and therefore we can write $f(T) \subseteq \bar{\kappa}(M) \subseteq \kappa(M) \subseteq \tilde{\kappa}(M) \subseteq f(T)$.

(i) $\Longrightarrow$ (iv) is obvious, (iv) $\Longrightarrow$ (v) by Proposition 9 and (v) $\Longrightarrow$ (vi) by Proposition 5.

(vi) $\Longrightarrow$ (vii). $T_\kappa \subseteq T_{\tilde{\kappa}}$ since $\kappa \leq \tilde{\kappa}$. Let $M \in T_{\tilde{\kappa}}$ and $\bar{\kappa}(M/\bar{\kappa}(M)) = N/\bar{\kappa}(M)$. Then $N \in T_\kappa = T_{\bar{\kappa}}$ since T_κ is closed under extensions, and consequently $N = \bar{\kappa}(M)$. Hence $M/\bar{\kappa}(M) \in F_{\bar{\kappa}} = F_\kappa = F_{\tilde{\kappa}}$, and so $\tilde{\kappa}(M) = M \subseteq \bar{\kappa}(M)$. Thus $M = \bar{\kappa}(M) \in T_\kappa$.

(vii) $\Longrightarrow$ (viii) by Proposition 8.

(viii) $\Longrightarrow$ (xi). Let $M \in R\text{-}mod$. In the exact sequence

$$0 \to \tilde{\kappa}(M)/\tilde{\kappa}(\tilde{\kappa}(M)) \to M/\tilde{\kappa}(\tilde{\kappa}(M)) \to M/\tilde{\kappa}(M) \to 0$$

the first and the third module belong to $F_{\tilde{\kappa}} = F_\kappa$ (since $\tilde{\kappa}$ is a radical), and therefore $M/\tilde{\kappa}(\tilde{\kappa}(M)) \in F_\kappa = F_{\bar{\kappa}}$. So $\tilde{\kappa}(M) = \tilde{\kappa}(\bar{\kappa}(M))$, that is, $\tilde{\kappa}(M) \in T_{\tilde{\kappa}} = T_\kappa$, and hence $\tilde{\kappa}(M) \subseteq \bar{\kappa}(M) \subseteq \kappa(M) \subseteq \tilde{\kappa}(M)$.

The other implications are either trivial or follow immediately from Propositions 8, 9.

<u>Corollary 11</u>. Let κ be a preradical. Then:

(i) If T_κ (F_κ) is closed under extensions, then $\bar{\kappa}$ ($\tilde{\kappa}$) is an idempotent radical.

(ii) $\tilde{\tilde{\kappa}}$ and $\bar{\bar{\kappa}}$ are idempotent radicals.

iii) $\bar{\varkappa} \subseteq \tilde{\bar{\varkappa}} \subseteq \bar{\tilde{\varkappa}} \subseteq \tilde{\varkappa}$.

iv) If $\varkappa$ is idempotent (if $\varkappa$ is a radical), then $\tilde{\varkappa} = \bar{\tilde{\varkappa}} = \tilde{\bar{\varkappa}}$ ($\bar{\varkappa} = \bar{\tilde{\varkappa}} = \tilde{\bar{\varkappa}}$) is an idempotent radical.

v) If both $T_{\varkappa}$ and $F_{\varkappa}$ are closed under extensions, then $\bar{\varkappa} = \tilde{\bar{\varkappa}} \subseteq \varkappa \subseteq \bar{\tilde{\varkappa}} = \tilde{\varkappa}$.

(vi) If $\bar{\tilde{\varkappa}} = \tilde{\bar{\varkappa}}$ and both $T_{\varkappa}$ and $F_{\varkappa}$ are closed under extensions, then $\varkappa$ is an idempotent radical.

<u>Proof</u>. (i) By Theorem 10(v) ((x)).

(ii) By (i) and by Propositions 8, 9.

(iii) The only non-trivial inclusion is $\tilde{\bar{\varkappa}} \subseteq \bar{\tilde{\varkappa}}$. However $\varkappa \subseteq \tilde{\varkappa}$ implies $\bar{\varkappa} \subseteq \bar{\tilde{\varkappa}}$ and, since $\bar{\tilde{\varkappa}}$ is a radical, Proposition 6 yields $\tilde{\bar{\varkappa}} \subseteq \bar{\tilde{\varkappa}}$.

(iv) Since $\varkappa$ is idempotent, we have $\bar{\tilde{\varkappa}} \subseteq \tilde{\varkappa} = \tilde{\bar{\varkappa}} \subseteq \bar{\tilde{\varkappa}}$ by Proposition 7.

Similarly, if $\varkappa$ is a radical.

(v) is obvious.

<u>Example 12</u>. Let $R = Z$ (the ring of integers), p be a prime and $\varkappa$ be a preradical defined by $\varkappa(G) = p.G \cap G[p]$. Then, as one may check easily, $\bar{\varkappa}(H) = 0$ and $\tilde{\varkappa}(H) = H$, H being the Prüfer p -group. Hence $\tilde{\bar{\varkappa}} \neq \bar{\tilde{\varkappa}}$.

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