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A NOTE ON NONISOMORPHIC STEINER QUADRUPLE SYSTEMS

C.C. LINDNER x) and T.H. STRALEY, Auburn

Abstract: Let (Q, q) and (V, v) be Steiner quadruple systems. In [1] J. Doyen and M. Vandensavel give conditions under which the $|V|$ mutually disjoint subsystems $(Q \times \{x\}, b)$ of the direct product $(Q \times V, b)$ can be unplugged and replaced with any collection of quadruple systems $(Q \times \{x\}, b(x))$ so that the only subsystems of order $|Q|$ of the resulting quadruple system are the quadruple systems $(Q \times \{x\}, b(x))$. Namely, if $|V| = 2$ and $|Q| \equiv 2$ or $10 \pmod{12}$, $|Q| \neq 2$. In this note we generalize this result to (V, v) contains no subsystem of order $|Q|$ and for any $n > 1$, n the order of a subsystem of (V, v), $|Q|/n \not\equiv 2$ or $4 \pmod 6$.

Key words: Steiner quadruple systems, nonisomorphic Steiner quadruple systems.

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1. Introduction. A Steiner quadruple system (or more simply a quadruple system) is a pair (Q, q) where Q is a finite set and q is a collection of 4-element subsets of Q (called blocks) such that any three distinct elements of Q belong to exactly one block of q. The number $|Q|$ is called the order of the quadruple system (Q, q). Hanani proved in 1960 that the spectrum for quadruple systems is

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the set of all positive integers $n \equiv 2$ or $4 \pmod 6$ [2]. If (Q, q) and (V, v) are quadruple systems and $(Q \times V, \mathscr{b})$ denotes their direct product, then for each x in V, $(Q \times \{x\}, \mathscr{b})$ is a subsystem of $(Q \times V, \mathscr{b})$ which is isomorphic to (Q, q). See [1] or [5]. It is well known that a subsystem of a quadruple system can be "unplugged" and replaced with any quadruple system on these same elements and the result is always a quadruple system. Since the subsystems $(Q \times \{x\}, \mathscr{b})$ are mutually disjoint we can independently replace each subsystem $(Q \times \{x\}, \mathscr{b})$ of $(Q \times V, \mathscr{b})$ by any quadruple system $(Q \times \{x\}, \mathscr{b}(x))$ and the result is still a quadruple system which we will denote by $(Q \times V, \mathscr{b}^*)$. It is of considerable interest to determine under what conditions for <u>every</u> collection of quadruple systems $(Q \times \{x\}, \mathscr{b}(x))$ the only subsystems of $(Q \times V, \mathscr{b}^*)$ of order $|Q|$ are the quadruple systems $(Q \times \{x\}, \mathscr{b}(x))$. (The reason being, of course, that t collections of $|V|$ quadruple systems of order $|Q|$ such that no two collections can be isomorphically paired gives t nonisomorphic quadruple systems of order $|Q|\,|V|$.) In [1] J. Doyen and M. Vandensavel give conditions under which this is the case. Namely, when $|V| = 2$ and $|Q| \equiv 2$ or $10 \pmod{12}$, $|Q| \neq 2$. In this note we generalize these conditions to cases where $|V| > 2$. The techniques used in this note are analogous to those developed by the authors in [3], [4], and [7].

2. Nonisomorphic Steiner quadruple systems. Let $(Q, \mathfrak{q})$ and (V, v) be quadruple systems and $(Q \times V, \mathfrak{b})$ their direct product. For each x in V let $(Q \times \{x\}, \mathfrak{b}(x))$ be a quadruple system. In view of the above remarks, if the $|V|$ mutually disjoint subsystems $(Q \times \{x\}, \mathfrak{b})$ are unplugged and replaced by the $|V|$ mutually disjoint quadruple systems $(Q \times \{x\}, \mathfrak{b}(x))$, the result is still a quadruple system which, as above, we will denote by $(Q \times V, \mathfrak{b}^*)$. We remark that the $|V|$ mutually disjoint quadruple systems $(Q \times \{x\}, \mathfrak{b}(x))$ are not necessarily related to the corresponding subsystem $(Q \times \{x\}, \mathfrak{b})$ nor to each other. This observation is crucial in what follows. Now let $(Q \times V, \mathfrak{b}^*)$ be the quadruple system constructed above and let $(T, \mathfrak{b}^*)$ be any subsystem of $(Q \times V, \mathfrak{b}^*)$. Set $V' = \{x \in V \mid (q, x) \in T\}$ and $T_x = \{q \in Q \mid (q,x) \in T\}$.

Lemma. If $(Q \times V, \mathfrak{b}^*)$, $(T, \mathfrak{b}^*)$, V' and T_u are as above, then $|T_x| = |T_y|$ for all $x, y \in V'$.

Proof. Let $x \neq y \in V'$ and let (s, x) be any element in T_x and (t, y) any element in T_y. For each element $(s', x) \in T_x$ there is exactly one element $(t', y) \in T_y$ such that $\{(s,x), (s',x), (t,y), (t',y)\} \in \mathfrak{b}^*$. However, if $s' \neq s$ then $t' \neq t$ so that $|T_x| \leq |T_y|$. A similar argument shows that $|T_y| \leq |T_x|$ so that $|T_x| = |T_y|$.

Theorem 1. Let $(Q \times V, \mathfrak{b}^*)$ be the quadruple system constructed above. Suppose that (V, v) contains no subsystem of order $|Q|$. If for any $n > 1$, where n is the order of a subsystem of (V, v), $|Q|/n \not\equiv 2$ or $4 \pmod 6$, then the only subsystems of $(Q \times V, b^*)$ of order $|Q|$ are the $|V|$ mutually disjoint quadruple systems $(Q \times \{x\}, b(x))$.

Proof. Let (T, b^*) be a subsystem of $(Q \times V, b^*)$ of order $|Q|$ and let $V' = \{x \in V \mid (q, x) \in T\}$. Since (V, v) contains no subsystem of order $|Q|$ it follows from the Lemma that $|T_x| = |T_y| = t \geq 2$ for all $x, y \in V'$. Hence $|T| = mt$ where $m = |V'|$. Since each of $(Q \times \{x\}, b^*)$ and (T, b^*) is a subsystem of $(Q \times V, b^*)$ and $T_x \times \{x\} = (Q \times \{x\}) \cap T$ we must have either $|T_x| = |T_x \times \{x\}| = 1$ or $|T_x| \equiv 2$ or $4 \pmod 6$. As $|T_x| \geq 2$ we must have $|T_x| \equiv 2$ or $4 \pmod 6$. Hence $|T|/m \equiv 2$ or $4 \pmod 6$. But (V', v) is a subsystem of (V, v) and so $|V'| = 1$. Hence $T = Q \times \{x\}$ for some x in V which completes the proof.

Let s and t be positive integers. We will denote by P_s^t the number of t-tuples of integers $(x_1, x_2, \dots, x_t)$ where $x_1 + x_2 + \dots + x_t = s$ and $0 \leq x_i < s$, $i = 1, 2, \dots, t$. The following theorem is the main result in this note.

Theorem 2. Let q and v be positive integers $\equiv 2$ or $4 \pmod 6$ and suppose there exists a quadruple system (V, u) of order v containing no subsystem of order q. If for any $n > 1$, where n is the order of a subsystem of (V, u), $|Q|/n \equiv 2$ or $4 \pmod 6$ then the construction in Theorem 1 gives at least P_v^t nonisomorphic

Steiner quadruple systems of order qv where t is the number of nonisomorphic quadruple systems of order q.

Remark. Note that if $|V| = 2$ and $|Q| \equiv 2$ or $10 \pmod{12}$, $|Q| \neq 2$, the conditions of Theorem 2 are automatically satisfied so that Theorem 2 is in fact a generalization of the result of Doyen and Vandensavel [1] mentioned in the introduction.

3. Example. Let $q = 14$ and $v = 4$. N.S. Mendelsohn and H.S.Y. Hung have shown that there are exactly 4 nonisomorphic quadruple systems of order 14 [6]. The only subsystems of a quadruple system of order 4 have orders 1, 2, or 4. Since neither $\frac{14}{2}$ nor $\frac{14}{4}$ is $\equiv 2$ or $4 \pmod{6}$, Theorem 2 gives at least $P_4^4 = 35$ nonisomorphic Steiner quadruple systems of order 56. As far as the authors can tell, this cannot be obtained via the results of Doyen and Vandensavel [1] since $56 = 28 \cdot 2$ and $28 \not\equiv 2$ or $10 \pmod{12}$.

The spectrum for pairs of nonisomorphic quadruple systems remains open.

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Mathematics Department
Auburn University
Auburn, Alabama 36830
U.S.A.

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