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ON EXTREMAL STRUCTURE OF WEAKLY LOCALLY COMPACT CONVEX SETS IN BANACH SPACES

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The note strengthens in some direction the results of Professor V. Klee concerning the extremal structure of locally compact convex sets, for the case of the weak topology of real Banach spaces.

Definition 1. A Banach space X is (LUR) if x_n, $x \in X$, $\|x_n\| = \|x\| = 1$, $\|x_n + x\| \longrightarrow 2$ imply $\|x_n - x\| \to 0$. X is (R) if all norm boundary points of its closed unit ball are its extreme points.

Definition 2 ([8]). A point x of a convex set C in a Banach space X is an exposed point of C if there is an $f \in X^*$ such that $f(y) < f(x)$, $\forall y \in C$, $y \neq x$. A point x of a convex C is a strongly exposed point of C if there is an $f \in X^*$ such that $f(y) < f(x)$, $\forall y \in C$, $y \neq x$, and moreover, whenever $f(y_n) \to f(x)$, $y_n \in C$, then $\|y_n - x\| \to 0$.

Definition 3. A Banach space X^* has (W^*S) property ((W^*) property) if any w^* compact convex subset of X^* is the w^* closed convex hull of its points that are

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strongly exposed (exposed) by functionals from X .

<u>Remark 1.</u> It follows from the results of J. Lindenstrauss and H.H. Corson ([8],p.142, [3],p.410 or [9], Th. 6.5) that any Banach space X with an equivalent (LUR) norm has the property that any weakly compact convex subset of X is the closed convex hull of its strongly exposed points.
Together with a very recent (LUR) -renorming theorem of S. Trojanski [11] for weakly compactly generated Banach spaces it means that every weakly compact convex subset of an arbitrary Banach space is the closed convex hull of its strongly exposed points. Furthermore, E. Asplund proved in [1],p. 46 that for any Banach space X such that there is an equivalent norm on X whose dual norm on X^* is (LUR), X^* has (W^*S) property.
Similarly, using the results of [1] and [2] , if X has an equivalent norm whose dual norm on X^* is (R) , then X^* has (W^*) property ([12], Th.2).

We will need the following two results of V. Klee:

<u>Theorem 1</u> (V. Klee,[4],p. 236). Suppose C is a locally compact closed convex subset of a locally convex Hausdorff linear space X and C contains no line. Suppose $0 \in C$ and K is the union of all closed half-lines which emanate from 0 and lie in C . Then X admits a continuous linear functional f which is positive on $K \setminus (0)$. For each such f and each real t , the set $C \cap f^{-1}(-\infty, t\rangle$ is compact.

<u>Theorem 2</u> (V. Klee [4],p. 237 or [6], p. 340). Assume

C is a locally compact closed convex subset of a locally convex Hausdorff linear space X, C contains no line. Then C has an extreme point.

<u>Definition 4</u>. A strongly exposed ray of a convex set C in a Banach space X is a closed halfline $h \subset C$ such that there is a closed supporting hyperplane H of C such that $H \cap C = h$ and moreover, whenever $\lim_{n \to +\infty} \varphi(x_n, H) = 0$, $x_n \in C$, $\{x_n\}$ bounded, then $\lim_{n \to \infty} \varphi(x_n, h) = 0$, where $\varphi(x, A)$ means the distance of x from the set A given by the norm of X.

<u>Remark 2</u>. If $H = \{x \in X ; f(x) = \gamma\}$, $f \in X^*$, $f \neq 0$, it is easy to see $\varphi(x, H) = |f(x) - \gamma| / \|f\|$ (see for instance [10],p. 21) and thus for a convex set C in a Banach space X a closed halfline $h \subset C$ is a strongly exposed ray of C iff there is an $f \in X^*$, $f \neq 0$, and a real γ such that $f(x) \leqq \gamma \ \forall x \in C$, $\{x \in X ; f(x) = \gamma\} \cap C = h$ and moreover, whenever $f(y_n) \rightarrow \gamma$, $y_n \in C$, $\{y_n\}$ bounded, then $\varphi(x_n, h) \rightarrow 0$.

Furthermore, it is easy to see that for instance the example of [8], p.145 of an exposed point of a bounded closed convex set which is not strongly exposed can easily produce an example of an exposed ray of a convex closed weakly locally compact set which is not a strongly exposed ray.

In the sequel, we will use the following notations:

<u>Notations</u>. Let X be a Banach space, $S \subset X$. Then

$w\,cl\,S$ (respectively $w^*cl\,S$) mean the weak (respectively the weak-star) closure of S in X. $cl\,con\,S$ resp. $w^*cl\,con\,S$ mean the norm closed convex hull resp. the weak-star closed convex hull of S in X. $\mathrm{Int}\,C$ resp. $\mathcal{B}(C)$ mean the norm interior resp. the norm boundary of $C \subset X$. If C is convex, $ext\,C$ resp. $exp\,C$ resp. $s\,exp\,C$ resp. $r\,exp\,C$ resp. $sr\,exp\,C$ mean the set of all its extreme points resp. exposed points resp. strongly exposed points resp. the set of all its exposed rays resp. strongly exposed rays. For a convex $C \subset X^*$, $exp_*\,C$ resp. $s\,exp\,C$ resp. $r\,exp\,C$ resp. $sr\,exp\,C$ mean the set of all its points that are exposed by functionals from X resp. strongly exposed by functionals from X resp. the set of all its exposed rays that are exposed by functionals from X resp. the set of all its strongly exposed rays that are strongly exposed by functionals from X. Furthermore, $[r\,exp\,C]$ denotes the union of all exposed rays of C and so on.

Now we may state our results that strengthen in some direction the results of V. Klee ([6], p. 91):

__Theorem 3__. Suppose X is a Banach space, $C \subset X$ is a closed convex weakly locally compact that contains no line. Then if $dim\,X > 1$

$$ext\,C \subset w\,cl(s\,exp\,C) \text{ and } C = cl\,conv((s\,exp\,C) \cup [sr\,exp\,C]).$$

__Theorem 4__. Assume X is a Banach space, C is a weakly-star closed convex weakly-star locally compact in X^* that contains no line. Then if $dim.\,X > 1$

(i) If X^* has (W^*) property, then $ext\, C \subset \subset w^*cl\, exp_* C$ and $C = w^*cl\, conv(exp_* C \cup [r\, exp_* C])$.

(ii) If X^* has (W^*S) property, then $ext\, C \subset \subset w^*cl\, s\, exp_* C$ and $C = w^*cl\, conv(s\, exp_* C \cup [s r\, exp_* C])$.

Proof. We will prove the part (ii) of Theorem 4. The other parts of Theorems 3,4 are proved similarly. We follow the ideas of the proof of Theorem 2.3 of V. Klee ([6],p. 91), only with some changes and additional considerations.
Take a $p \in ext\, C$ (see Theorem 2). Let K be the union of all closed halflines in C which emanate from p. Suppose $K \neq \emptyset$, since otherwise K is w^* compact, by the result of V. Klee (see for instance [7],p. 340). Using Theorem 1, take an $f \in X$, such that $f(x) > f(p)\ \forall x \in K$, $x \neq p$, and such that $\forall t$ real, $C \cap f^{-1}(-\infty, t\rangle$ is w^* compact. Choose an arbitrary $\gamma > f(p)$. Then $C \cap \cap f^{-1}(-\infty, \gamma+1\rangle$ is w^*-compact and therefore is the w^* closed convex hull of those of its points that are strongly exposed by functionals from X, by our hypotheses. Thus, by the Milman's theorem ([7], p. 332), $p \in w^*cl\, s\, exp_*(C \cap f^{-1}(-\infty, \gamma+1\rangle)$. Therefore, for an arbitrary w^*-neighborhood V of p, there is a point p_1 of the set $V \cap C \cap f^{-1}(-\infty, \gamma+1\rangle$ which is a strongly exposed point of $C \cap f^{-1}(-\infty, \gamma+1\rangle$ by some $g \in X$ and such that $f(p_1) < \gamma$.

To see p_1 is a strongly exposed point of C by $g \in \in X$, it suffices to show $g(x) \leqq g(p_1)\ \forall x \in C$ and whenever $g(x_m) \to g(p_1)$, $x_m \in C$, $m = 1, 2, \dots$, then $\| x_m - p_1 \| \to 0$. Let $H = C \cap f^{-1}(\gamma+1)$. Then

$H \neq \emptyset$. Take $x \in C$, $f(x) > r + 1$. Thus, obviously, the segment $\langle p_1, x \rangle$ crosses H at a point $x_1 \neq p_1$. If $g(x) > g(p_1)$, then $g(x_1) > g(p_1)$ - a contradiction. Now suppose there are a norm neighborhood $\mathcal{U}$ of p_1 such that $\mathcal{U} \subset f^{-1}(-\infty, r)$ and $x_n \in C$, $f(x_n) > r + 1$, $n = 1, 2, \dots$, such that $g(x_n) \longrightarrow g(p_1)$. Let $x_n^1 = \langle p_1, x_n \rangle \cap H$. Then $g(x_n^1) \in \langle g(x_n), g(p_1) \rangle$ and thus $g(x_n^1) \longrightarrow g(p_1)$. Furthermore, $x_n^1 \in C \cap f^{-1}(-\infty, r+1\rangle$ and $x_n^1 \notin \mathcal{U}$ -; a contradiction.

Now, denote by $A = w^* cl\, con\, (s\, exp_* C \cup [s\kappa\, exp_* C])$. Suppose $A \neq C$. Then there are an $F \in X$ and a real k such that $k \in FC$, $k + 1 < inf\, FA$. Assume without loss of generality $k = 0$ (otherwise take a suitable translation). Let $B = C \cap F^{-1}(-\infty, 1\rangle$. Then obviously $T = ext\, B \cap F^{-1}(-\infty, 1) \subset ext\, C$, and by the preceding part of the proof, $ext\, C \subset w^* cl\, s\, exp_* C$. Thus, if $T \neq \emptyset$ and $t \in T$, then $t \in A$, a contradiction. Thus (see Theorem 2), $\emptyset \neq ext\, B$ and $ext\, B \subset F^{-1}(1)$. By the preceding part of the proof, let $y \in s\, exp_* (C \cap F^{-1}(0))$. Then $y \notin ext\, B$ and thus there is an $x \in C$ such that $F(x) < 0$ and $\langle x, 2y - x \rangle \subset B$. Let $D = \{t ; y + t(x - y) \in C\}$. Then it is easy to see the endpoints of $y + D(x - y)$ lie in $ext\, C$. If $s = sup\, D < \infty$, then $y + s(x - y) \in ext\, B \setminus F^{-1}(1)$, a contradiction. Thus $sup\, D = +\infty$ and since C contains no line, $t = inf\, D > +\infty$. We show $h = y + D(x - y) \in s\kappa\, exp_* C$. Since y is an element of

$\mathrm{sexp}_*(C \cap F^{-1}(0))$, there is a w^* closed hyperplane H' in X^* such that H' supports the set $C \cap F^{-1}(0)$, $H' \cap F^{-1}(0) \cap C = \{y\}$ and, moreover, if $x_n \in F^{-1}(0) \cap C$, $\varphi(x_n, H') \to 0$, then $\|x_n - y\| \to 0$. If $F^{-1}(0) \cap C \neq \{y\}$, then obviously $H' \neq F^{-1}(0)$. If $F^{-1}(0) \cap C = \{y\}$, then we can choose such H' with the above properties again so that $H' \neq F^{-1}(0)$. Therefore suppose $H' \neq F^{-1}(0)$. Take $H = H' \cap F^{-1}(0)$. Then the codimension of H in X^* is 2, H is w^* closed in X^*, supports $F^{-1}(0) \cap C$ at y in $F^{-1}(0)$ and whenever $y_n \in F^{-1}(0) \cap C$, $\varphi(y_n, H) \to 0$, then a fortiori $\varphi(y_n, H') \to 0$ and thus $\|y_n - y\| \to 0$. Take now $J = H + R(x - y)$, where R denotes the reals. Then J is a weakly-star closed hyperplane in X^* (c.f.[4], p. 29). Let $y_1 = 2y - x$. If some $c \in C$, $F(c) < 0$ lies in the other open halfspace determined by J than is the closed halfspace in which $C \cap F^{-1}(0)$ is contained, then $\langle c, y_1 \rangle \cap F^{-1}(0)$ lies also in this open halfspace, a contradiction. Similarly for the case $F(c) > 0$ (taking x instead of y_1). Let now $J = \{x \in X^* ; G(x) = \gamma$, γ the real, $0 \neq G \in X\}$. Then $H = \{x \in F^{-1}(0) ; G(x) = \gamma\}$, and for

$$z \in F^{-1}(0), \ \varphi(z, J) = \frac{|G(z) - \gamma|}{\|G\|_{X^*}}, \ \varphi(z, H) = \frac{|G(z) - \gamma|}{\|G\|_{F^{-1}(0)}},$$

where $\|G\|_{F^{-1}(0)}$ means the common used supremum norm of $G \in (F^{-1}(0))^*$. Thus for $x_n \in F^{-1}(0)$, $\varphi(x_n, J) \to 0$ iff $\varphi(x_n, H) \to 0$. Now if for some $x_n \in C$, $\{x_n\}$ norm bounded, $F(x_n) \le 0$, $\varphi(x_n, J) \to 0$, then denoting by $x_n^1 = \langle x_n, y_1 \rangle \cap F^{-1}(0)$, it is easy to see $\varphi(x_n^1, J) = \varphi(x_n, J) \cdot \varphi(x_n^1, y_1) \cdot \varphi(x_n, y_1)^{-1}$. Thus $\varphi(x_n^1, J) \le \varphi(x_n, J)$ and $\varphi(x_n^1, H) \to 0$. Therefore $\varphi(x_n^1, h) \le \varphi(x_n^1, y) \to 0$. Now, again

$\varphi(x_n, h) = \varphi(x_n^1, h) \cdot \dfrac{\varphi(x_n, y_1)}{\varphi(x_n^1, y_1)}$, and

$0 < \delta \le \varphi(x_n^1, y_1) \le \varphi(x_n, y_1) \le K > 0$, $K < \infty$.

Thus $\varphi(x_n, h) \to 0$. Similarly for the case $x_n \in C$, $F(x_n) > 0$ (taking again x instead of y_1). Thus $A \not\ni h \in sn\, exp_* C$, a contradiction, which completes the proof.

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