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MORSE-SARD THEOREM FOR REAL-ANALYTIC FUNCTIONS

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In this paper we will prove that the set of all critical values must be countable for every real-analytic function, which is defined on $D \subset E_N$.

Definition 1. A real-valued function $f(x)$ defined on an open subset $D \subset E_N$ is called real-analytic, if each point $w \in D$ has an open neighborhood $\mathcal{U}$, $w \in \mathcal{U} \subset D$ such that the function has a power series expansion in $\mathcal{U}$.

Theorem 1. Let f be a real-analytic function defined on an open subset $D \subset E_N$. Let us denote by Z the set of critical values of f , i.e.

$$Z = \{ x \in D ; \frac{\partial f}{\partial x_i}(x) = 0 , \quad i = 1, 2, \dots , N \} ;$$

then the set $f(Z \cap K)$ is finite for every compact subset $K \subset D$ and hence $f(Z)$ is at most countable.

Remark. The Morse-Sard theorem for C^∞ -functions gives us only

$$H_\alpha(f(Z)) = 0$$

for all $\alpha > 0$ (where H_α is the α-dimensional Hausdorff measure). But we can construct an uncountable subset

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$M \subset E_1$ such that $H_\alpha(M) = 0$ for all $\alpha > 0$. On the other hand, there can be easily constructed a real-analytic function defined on $(0, 1)$ such that the set $f(Z)$ is infinite.

The proof of Theorem 1 is based on some theorems about germs of varieties from the theory of several complex variables. We recapitulate for the reader the necessary definitions and theorems from [G-R] in § 1.

§ 2 contains then the proof of Theorem 1.

§ 1. Germs of varieties

This paragraph is only a recapitulation of the facts from [G-R] (in brackets we shall refer to the numbers of definitions and theorems in [G-R]).

Definition 2 (II.E.4). Let X, Y be subsets of $\mathbb{C}^N$ (the Cartesian product of N copies of the complex plane). The sets X and Y are said to be equivalent at 0 if there is a neighborhood $\mathcal{U}$ of 0 such that $X \cap \mathcal{U} = Y \cap \mathcal{U}$. An equivalence class of sets is called the germ of a set. The equivalence class of X is to be denoted by $\mathcal{X}$.

If $\mathcal{X}_1, \mathcal{X}_2$ are germs of a set, we can define $\mathcal{X}_1 \cup \mathcal{X}_2$, $\mathcal{X}_1 \cap \mathcal{X}_2$ by the natural way.

Definition 3 (II.E.6). A germ $\mathcal{X}$ is the germ of a variety if there are a neighborhood $\mathcal{U}$ of 0 and functions $f_1, \dots, f_t$ holomorphic in $\mathcal{U}$, such that

$$\{x \in \mathcal{U} ; f_i(x) = 0, \quad 1 \le i \le t\}$$

is a representative for $\mathcal{X}$.

We shall denote the collection of germs of a variety at 0 by $\mathcal{B}$.

<u>Definition 4</u> (II.E.12). A germ $V \in \mathcal{B}$ is said to be <u>irreducible</u> if $V = V_1 \cup V_2$ for $V_1, V_2 \in \mathcal{B}$ implies either $V = V_1$ or $V = V_2$.

<u>Theorem 2</u> (II.E.15). Let $V \in \mathcal{B}$. We can write $V = V_1 \cup \dots \cup V_k$ where the V_i are irreducible and $V_i \not\subset V_j$ for $i \neq j$. $V_1, \dots, V_k$ are uniquely determined by V .

An open polydisc in $\mathbb{C}^N$ is a subset $\Delta(w, r) \subset \mathbb{C}^N$ of the form

$$\Delta(w, r) = \Delta(w_1, \dots, w_N ; r_1, \dots, r_N) =$$

$$= \{ z \in \mathbb{C}^N ; |z_j - w_j| < r_j , \ 1 \leq j \leq N \} .$$

<u>Definition 5</u> (I.B.8, I.B.10). A subset M of $\mathbb{C}^N$ is a complex submanifold of $\mathbb{C}^N$ if to every point $p \in M$ there correspond a neighborhood $\mathcal{U}$ of p , a polydisc $\Delta(0, \sigma)$ in $\mathbb{C}^k$ $(k \leq N)$ and a nonsingular holomorphic mapping $F : \Delta(0, \sigma) \longrightarrow \mathbb{C}^N$ such that $F(0) = p$, and

$$M \cap \mathcal{U} = F(\Delta(0, \sigma)) .$$

<u>Theorem 3</u>. Let $V \in \mathcal{B}$ be an irreducible germ. Then there exist a polydisc $\Delta(0, r)$ and a set $V_0 \subset \Delta(0, r)$ such that:

(i) $\overline{V_0}$ is a representative of V ,

(ii) for each polydisc $\Delta_1(0) \subset \Delta$ there exists a polydisc $\Delta_2(0) \subset \Delta_1(0)$ such that $V_0 \cap \Delta_2$ is a connected complex submanifold.

This theorem follows immediately from III.A.10, III.A.9 and III.A.8; this is only a reformulation of a part of Theorem III.A.10.

§ 2. The proof of Theorem 1

Let $x_0 \in D$ be fixed. Suppose that there exist points $x_n \in D$ such that

(1) $x_n \longrightarrow x_0$,

(2) $\operatorname{grad} f(x_n) = 0$, $n = 1, 2, \ldots$,

(3) if $n \neq m$ then $f(x_n) \neq f(x_m)$.

We want to show that such sequence cannot exist.

Suppose that $x_0 = 0$ (for easy notation). In a small neighborhood of the point 0 we can write

$$f(x) = \sum_{\alpha_1, \ldots, \alpha_N \geq 0} a_{\alpha_1, \ldots, \alpha_N} x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_N^{\alpha_N} .$$

We can consider $E_N \subset \mathbb{C}_N$ and extend the function f on a small polydisc $\Delta = \Delta(0, r) \subset \mathbb{C}^N$:

$$f(z) = \sum_{\alpha_1, \ldots, \alpha_N \geq 0} i\, a_{\alpha_1, \ldots, \alpha_N} z_1^{\alpha_1} \cdot z_2^{\alpha_2} \ldots z_N^{\alpha_N} \; ; \; z \in \Delta(0, r) .$$

From (2) we have (if $x_n \in \Delta(0, r)$)

$$\frac{\partial f}{\partial z_i}(x_n) = 0 , \quad i = 1, \ldots, N .$$

Let $\mathbb{V} \in \mathcal{B}$ be the germ of a variety determined by the set

(4) $V = \{ z \in \Delta(0, r) \; ; \; \dfrac{\partial f}{\partial z_1}(z) = 0, \ldots, \dfrac{\partial f}{\partial z_N}(z) = 0 \}$.

There is a decomposition $\mathbb{V}$ into its irreducible branches (see Theorem 2)

$$\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2 \cup \dots \cup \mathbb{V}_k .$$

If $V_1, \dots, V_k$ are representatives of $\mathbb{V}_1, \dots, \mathbb{V}_k$ then there exists a polydisc $\Delta_1(0)$ such that

(5) $$V \cap \Delta_1 = (V_1 \cap \Delta_1) \cup \dots \cup (V_k \cap \Delta_1) .$$

By (1) we have (for all n sufficiently large)

$$x_n \in V \cap \Delta_1$$

and hence infinite number of x_n must lie in some $V_i \cap \cap \Delta_1$. So we can suppose that there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that

(6) $$x_{n_j} \in V_1 \cap \Delta_1$$

for all j. Because the germ $\mathbb{V}_1$ is irreducible, it follows from Theorem 3 that there exist a polydisc $\Delta_2(0)$ and a set $V_0 \subset \Delta_2$ such that

(i) $\overline{V_0}$ is a representative of $\mathbb{V}_1$,

(ii) for every polydisc $\Delta_3(0) \subset \Delta_2$ there exists a polydisc $\Delta_4(0) \subset \Delta_3$ such that $V_0 \cap \Delta_4$ is a connected complex submanifold.

Because the sets V_1 and $\overline{V_0}$ are both representatives of the same germ $\mathbb{V}_1$, there exists a polydisc $\Delta_3(0)$ such that

$$V_1 \cap \Delta_3 = \overline{V_0} \cap \Delta_3 .$$

There exists (by (ii)) a polydisc $\Delta_4(0) \subset \Delta_3 \cap \Delta_1$ such that

(7) $$V_1 \cap \Delta_4 = \overline{V_0} \cap \Delta_4$$

and $V_0 \cap \Delta_4$ is a connected complex submanifold.

We shall prove that f must be constant on $\overline{V_0} \cap \cap \Delta_4$. Let $x_0 \in V_0 \cap \Delta_4$ be fixed, let us denote

$$M = \{x \in V_0 \cap \Delta_4 ; f(x) = f(x_0)\} .$$

Suppose $x \in M$. By Definition 5 there exist a neighborhood $\mathcal{U}$ of x, a polydisc $\Delta_k \subset \mathbb{C}^k$, $(k \leq N)$ and a nonsingular holomorphic mapping F :

$$F : \Delta_k \longrightarrow \mathbb{C}^N$$

such that

$$F(\Delta_k) = \mathcal{U} \cap V_0 \; ; \; F(0) = x .$$

Hence for arbitrary $w \in \mathcal{U} \cap V_0$ there exists $p \in \in \Delta_4$ such that

$$F(p) = w .$$

Let us denote

$$\gamma(t) = tp ; \quad 0 \leq t \leq 1 .$$

Then $F(\gamma(t))$, $0 \leq t \leq 1$ is a smooth curve, lying in $\mathcal{U} \cap V_0$ and by (7) and (5) we have

$$F(\gamma(t)) \in V , \quad 0 \leq t \leq 1 ,$$

and hence (by (4))

$$\frac{d}{dt}[f(F(\gamma(t))] = 0 , \quad 0 \leq t \leq 1 .$$

From this it follows immediately that $f(w) = f(x)$, hence

$$\mathcal{U} \cap V_0 \subset M .$$

Because the set M is open and closed in $V_0 \cap \Delta_4$, we have

$$V_0 \cap \Delta_4 = M .$$

The function f is a constant function on $V_0 \cap \Delta_4$, and hence also on $\overline{V_0} \cap \Delta_4$. But from (1),(6),(7) we have (for $j \geq j_0$)

$$x_{n_j} \in \overline{V_0} \cap \Delta_4 ,$$

hence $f(x_{n_j}) = f(x_{n_\ell})$; $\ell, j \geq j_0$, which is a contradiction with (3).

Now the proof of Theorem 1 can be easily finished. Suppose that $K \subset D$ is a compact set and that the set $f(Z \cap K)$ is intinite. We can find a sequence $\{x_n\}^{\infty} \subset$ $\subset Z \cap K$ such that $f(x_n) \neq f(x_m)$ (for $n \neq m$). Then there exists a subsequence $\{x_{n_k}\}$, $x_{n_k} \rightarrow x_0 \in K$. Because (1),(2),(3) is true for $\{x_{n_k}\}$, we have a contradiction.

R e f e r e n c e

[G-R] R.GUNNING, H. ROSSI: Analytic functions of several complex variables, Prentice-Hall,1965.

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