Commentationes Mathematicae Universitatis Carolinae

13,4 (1972)

EITHER TOURNAMENTS OR ALGEBRAS ?

Vladimír MÜLLER, Jaroslav NEŠETŘIL, Jan PELANT, Praha

(Preliminary communication)

Introduction. The program of a systematic study of algebraic properties of graphs and relations in general was carried out by K. Čulík, G. Sabidussi, Z. Hedrlín and A. Pultr. While this approach led to an undoubtedly worthy success in applications of the graph theory to various branches of mathematics (see [2]) within the graph theory itself, the role of this approach is still discussed.

Certainly there are parts of the graph theory where the study of properties of graphs by means of homomorphisms between them is generally known (chromatic numbers and polynomials). But this being as far more an exception than a rule, it is not much surprising that there are graphs - namely tournaments - which are basically the same as the algebras of certain kind but which have not been yet studied from this point of view. (As far as we know, [1] is the only paper dealing with the subject, except, of course, the work done on the study of the automorphism group of a tournament, see [3]). In 1965, Z. Hedrlín observed the inversion of a tournament $\mathcal{T} = (T, t)$ into the

---

AMS, Primary: 05C20, 05C25 Ref.Ž. 8.83

algebra $A_{\mathcal{T}} = \langle T, \cdot \rangle$ defined by $x \cdot y = y \Longleftrightarrow (x,y) \in t$. This correspondence between the class of all tournaments and the class of all commutative groupoids $\langle X, \cdot \rangle$ satisfying $x \cdot y \in \{x, y\}$ for all $x, v \in X$ is clearly a bijection. A moment of insight is enough to check that the tournament-homomorphisms and the algebraic homomorphisms are also in 1-1 correspondence and actually coincide. It was our aim to go on in this direction and to study tournamensts in an algebraic way. [1)]

Basic concepts. We refer to [3] for the notions not defined here. A tournament $\mathcal{T} = (T, t)$ is a finite set T with a relation $t \subset T \times T$ which is reflexive and satisfies $(x,y) \in t \Longleftrightarrow (y,x) \notin t$ for any two distinct vertices of T. Let $\mathcal{T} = (T, t)$ and $\mathcal{S} = (S, s)$ be tournaments. A mapping $f: T \longrightarrow S$ is called a homomorphism if $(f(x), f(v)) \in s$ whenever $(x,y) \in t$. In an obvious sense we use the terms endomorphism, automorphism, isomorphism. Denote by $(E(\mathcal{T}), \wedge, \vee)$ the lattice of all congruences of $\mathcal{T}$ (in the algebraic sense). The tournaments with the simplest congruence lattice are of special importance. E.C. Milner proposed to call them simple (while we originally used the term "fragile", we shall adopt Milner´s notation). The main role of simple tournaments is in the simple decomposition:

----

1) This paper contains some of the results achieved at the seminar on Graph Theory 1970-71 at the Charles University, Prague. under guidance of Z. Hedrlín and the second author.

Proposition. Let $\mathcal{T} = (T, t)$ be a strong tournament. Then there exists the unique congruence η on $\mathcal{T}$ such that $\mathcal{T} | \eta$ (the factor tournament) is a simple tournament with at least two vertices.

Thus with the aid of the simple decomposition and using the fact that every homomorphism onto is a retraction, we can reduce a strong tournament to another, hopefully simpler tournament, but anyhow we lessen the number of vertices. It is possible to say that the cyclic decomposition of non-strong tournaments and the simple decomposition of strong tournaments are the two tools on reduction of tournaments.

This reduction plays an important role in proofs of our theorems. (Of course, asymptotically most of the tournaments are simple tournaments.)

Some of the results.

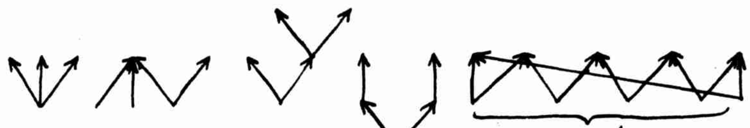
1. Characterization of congruence lattices:

Let $(L, \wedge, \vee)$ be a lattice. Denote by $(I(L), \geq)$ the set of all irreducible elements of L.
Let $(O, \geq)$ be a partial ordering. Put: $m(O) = (O, R)$ where the relation R is defined by $(x, y) \in R$ iff $x \leq y$ and there exists no $z \in X$ such that $x < z < y$;
$max(O) = \{y \in X \mid y \leq x \Longrightarrow x = y\}$; $max^{-1}(O) = \{y \in X \mid y \not\leq x \Longrightarrow x \in max(O)\}$.

Theorem. Let L be a finite lattice. Then L is a lattice isomorphic to $E(\mathcal{T})$ for a tournament $\mathcal{T}$ iff

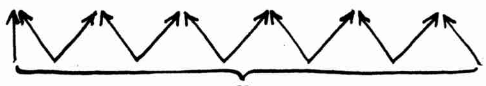
1) L is distributive,

2) $m(I(L))$ does not contain any of the following subgraphs:



$n \geq 1$

3) Either $|max(I(L))| = 1$ or the partial ordering induced by $m(I(L))$ on $max(I(L)) \cup max^{-1}(I(L))$ have to be isomorphic to a graph which arises from the graph $(n > 1)$



n

by deleting some of the lower vertices.
A finite lattice satisfying 1), 2), 3) is said to be admissible. Investigating the simple tournaments we are able to prove:

<u>Main theorem</u>. Let L be an admissible lattice, let G be a finite group of odd order. Then there exists a tournament $\mathcal{T}$ with $A(\mathcal{T}) \simeq G$ and $E(\mathcal{T}) \simeq L$.

2. Forcing of homomorphisms by other properties of tournaments.

It is an easy modification of the main theorem that, given an admissible lattice L , an even group G and a tournament $\mathcal{S}$, there exists a tournament $\mathcal{T}$ such that $\mathcal{S}$ is a subtournament of $\mathcal{T}$ and the statement of the theorem still holds. Hence with respect to the endomorphisms and automorphisms there are locally no "forbidden parts". In a way this holds globally, too.

We call the set of the degrees of all vertices of a tournament $\mathcal{T}$ the degree sequence (the score vector) of $\mathcal{T}$. The following is true:

Theorem. For any degree sequence which is strong with at least 5 vertices there exists a simple tournament.

Theorem. For any degree sequence which is strong with at least 6 vertices there exists a tournament with the trivial group of automorphisms.

(Both theorems have the full version which gives the complete solution.)

Thus one cannot force any non-trivial endomorphism or automorphism by a degree sequence. The forcing of the simplicity, identity and other related problems will be discussed in a forthcoming paper which is going to appear under the title of this communication.

3. S+H-tournaments.

A tournament is said to be homogeneous if its degree sequence is constant. We consider more "regular" tournaments, we call them strongly homogeneous tournaments (briefly S+H-tournaments): A tournament $\mathcal{T}$ is an S+H-tournament if $|\{z \mid (x, y) \in t$ and $(y, z) \in t\}| = const$ for any two distinct vertices of $\mathcal{T}$. We prove:

First characterization of S+H tournaments:

$\mathcal{T} = (T, t)$ is an S+H-tournament iff $\mathcal{T}$ is homogenous and $\mathcal{T}|_{v(a)}$ is homogeneous for every $a \in T$ ($v(a) = \{x \mid (a,x) \in t\}$).

Second characterization of S+H tournaments:

Let $\mathcal{T}$ be a tournament, let t be not an ordering of T. Then the following two statements are equivalent:

1) $\mathcal{T}$ is an S+H-tournament,

2) $\mathcal{T}|_{T \setminus \{x,y\}} \sim \mathcal{T}|_{T \setminus \{u,v\}}$ for any pairs of distinct vertices x, y and u, v of T. (We write $\mathcal{T} \sim \mathcal{S}$ if $\mathcal{T}$ and $\mathcal{S}$ have the same degree sequences.)

Third characterization of S+H-tournaments:

$\mathcal{T}$ is an S+H-tournament iff $s(\mathcal{T}) = \frac{T-1}{2}$.

Here the number $s(\mathcal{T})$ is the least number $|M|$ where $M \subset t$ is a set for which the tournament $(T, (t \setminus M) \cup M^{-1})$ is not simple.

We prove that every S+H-tournament with n vertices can be extended to an S+H-tournament with $2n+1$ vertices and this is the smallest extension. (With the help of this construction we prove that for every n there exists an n-universal S+H-tournament.) We give another recursive construction of S+H-tournaments: Given $\mathcal{T}$, $\mathcal{S}$ -S+H-tournaments we can construct an S+H-tournament with $(|T|+1)(|S|+1)-1$ vertices providing that the tournament $\mathcal{S}$ is invertible (= there exists a bijection $\varphi : S \longrightarrow S$ such that $(x,y) \in s$ iff $(\varphi(x), \varphi(y)) \notin s$) .

The S+H-tournaments are naturally connected with two kinds of block designs, one of them being Hadamard block designs. Using a standard tehnique on difference sets (with minor

modifications) we can construct an invertible S+H-tournament on every prime power set.
This is going to appear in Czech.Math.J. in a paper by the first and third authors of this communication. A related paper of the same authors entitled "Depth and simplicity" will appear shortly in this journal.

Remark: Our main theorem gives a sharpening of the Moon's result [3]. On the other hand it presents a type of questions considered in the universal algebra. (Only recently W. Lampe proved the independency of congruence lattices and groups of automorphisms of universal algebras.)

R e f e r e n c e s

[1] V. CHVÁTAL: On finite countable rigid graphs and tournaments, Comment.Math.Univ.Carolinae 6(1965), 429-438.

[2] Z. HEDRLÍN: Extensions of structures and full embeddings of categories, Actes Congrès intern.math. 1970,T1,319-322.

[3] J.W. MOON: Topics on Tournaments, Holt,Rinehart and Winston,New York 1968.

Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, Praha 8
Československo

(Oblatum 13.11.1972)